

**SOLID
GEOMETRY**

—
AVERY

~~A. Smith~~

R. Leonard

to Mathematics Dept.

SOLID GEOMETRY

BY

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PREFACE

SOLID GEOMETRY should be the most interesting and fascinating branch of mathematics studied in secondary schools. To make it so this text treats it, not as a mere supplement to plane geometry, but as a subject distinct. The aim is to develop in the pupil greater facility in visualizing spatial relations and figures, in representing such figures, and in solving problems concerning them.

A pupil beginning the study of solid geometry has very little mathematical imagination. All figures appear to him to be two-dimensional and he finds the visualizing of space relations difficult. Every possible aid must be given in forming his early space concepts.

For this reason the introduction of this text endeavors by questions on the relationships of points, lines, and planes in space, to develop in the pupil the ability to see fundamental relations and to represent them on paper. Without this ability he cannot understand solid geometry, but will be forced to memorize the text.

The order of theorems beginning Book I and the treatment of Proposition III are due to suggestions made by pupils. The common complaint has been: "Why are we given the most difficult theorem first without any preparation for it?"

The *plan of attack* used in the author's plane geometry is continued. Consistently used, it overcomes the tendency to memorize the text-proved theorems and creates the confidence necessary for the solution of the exercises.

Throughout the text emphasis is placed upon certain leading propositions. These include the starred theorems of the College Entrance Board's list and the Fundamental Theorems of the

National Committee Report. Many theorems formerly included are so placed that they may be omitted in a minimum course when time is limited.

Books II and III include numerous exercises in computation, based upon the formulas established. Formal proofs involving the theory of limits are replaced by informal proofs based upon inference. Cavalieri's Theorem and the Prismatoid Formula are given, and suggestions are made effecting desirable simplification and generalization in the treatment of mensuration theorems.

In order that the better students may not be deprived of the pleasure of discovering the solution of exercises, figures and helps are not given with the exercises. In the Appendix the pupil may find "Suggestions and Helps" for the solution of the more difficult exercises.

R. A. A.

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SYMBOLS

$\angle$	angle	$>$	is greater than
$\triangle$	triangle	$<$	is less than
$\square$	parallelogram	$\dots$	and so on
$\perp$	perpendicular, <i>or</i> is perpendicular to	$($	arc
$\parallel$	parallel, <i>or</i> is parallel to	$\sim$	similar, <i>or</i> is similar to
$=$	equals, is equal to, <i>or</i> is congruent to	$\therefore$	therefore
$=^{\circ}$	equals in degrees	$\odot$	circle
		A', B', C'	A prime, B prime, C prime

SOLID GEOMETRY

INTRODUCTION

1. Solid Geometry. — In our work in plane geometry we dealt with figures of only two dimensions, length and breadth. Solid geometry introduces us to figures which require a third dimension, thickness (height or depth). In plane geometry we learned about points, lines, angles, polygons, and circles, — figures all points of which lie in the same plane.

Now we are to study pyramids, cones, spheres, cylinders, and other solids, — figures whose parts do not all lie in the same plane. Material objects representing these figures surround us everywhere, so we may think of our study of solid geometry as having many concrete applications. We are already familiar with many of these figures: we have seen pictures of the pyramids of Egypt; we have played baseball with a sphere; and we know how many cylinders our automobile has.

2. Geometric Solids. — But in solid geometry we need to think of these solids in a new way. If we picture in our minds the portion of space which a material solid occupies, rather than the object itself, we are thinking of a *geometric solid*. The boundaries of this solid are called *surfaces*. When a workman pours concrete into a mold, the resulting block is a material solid; the hole, or space left when the block is removed, represents a geometric solid. A *geometric solid* is a three-dimensional figure; that is, a figure not all parts of which lie in the same plane.

3. Representing the Figures of Solid Geometry. — Since the paper and blackboard upon which we are to represent the figures of solid geometry are plane surfaces, or two-dimensional, it may be difficult at the start to visualize a three-dimensional figure when thus representing it. The process is somewhat similar to

that which an architect must follow when he designs a building. He not only plans the size and shape of the various parts, but also pictures the completed structure. The floor plans, detail, and elevation drawings which he makes are but representations, on the surface of the paper, of his mental picture. Others working from these plans must be able to see his vision and reproduce it in material form. To work with geometric solids we must develop this ability to see figures in three dimensions. For this reason let us investigate and represent some of the relations existing between lines and planes.

The ceiling and side walls of this room are planes which meet in a corner forming three lines, each of which is perpendicular to the other two. Make a drawing to represent the upper right corner of the room showing these lines and the three right angles formed. Make a similar drawing to represent the corner formed by two adjacent side walls and the floor. You thus have drawn a line perpendicular to each of two other lines at their point of intersection. Do these lines in the drawing seem to you to be perpendicular? Why? Is it possible in plane geometry to have two lines perpendicular to the same line at the same point?



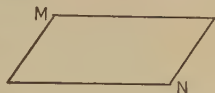
Preliminary Study of Lines and Planes

If we keep in mind that solid geometry deals with figures whose parts are not all in the same plane, we now are ready to investigate some of the relations that may exist between lines and planes. To do so, let us remember that a line is unlimited in length and that a plane is unlimited in extent. In our study we may use any of the facts learned in plane geometry about plane figures. But first we must know definitely what a plane is and how a plane is determined.

4. A Plane. — A *plane* is a surface, in which, if any two points are taken, the straight line that joins them lies wholly in the surface.

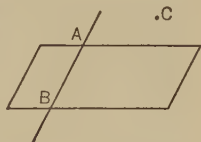
A plane usually is represented by a parallelogram or by a trapezoid; as the plane MN .

A plane is determined by certain points and lines, if that plane and no other plane contains those points and lines.



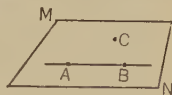
Since it is evident that a plane through the line AB , if rotated about the line, can be made to pass through a given point C in only one position, we may assume the following statement:

A plane is determined by a straight line and a point not in the line.

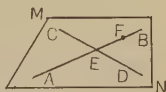
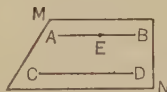


Other methods of determining a plane, dependent upon this statement, are:

A plane is determined by any three points not in the same straight line.



A plane is determined by two parallel lines, or by two intersecting lines.



5. Two Lines in Space. — Two lines may be placed so that no plane is determined by them. In plane geometry two lines either intersect or are parallel. In solid geometry two lines which do not meet need not be parallel; they may not be in the same plane. So we must remember that two lines in space may be in the same plane, hence parallel or intersecting; or they may be in different planes.

When using the definition of parallel lines in plane geometry we *assume* that the lines are in the same plane. In solid geometry before we can prove that two lines are parallel we *must show* that the lines are in the same plane.

Lines which are not in the same plane and hence cannot meet are called *skew* lines; or, simply, lines which are not parallel and do not meet are called skew lines.

By referring to the lines formed by the ceiling, side walls,

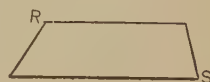
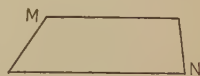
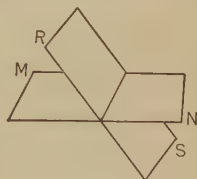
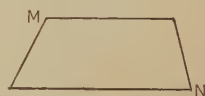
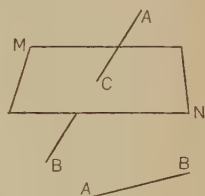
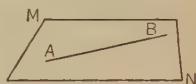
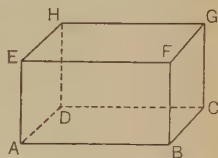
and floor, locate two lines which intersect; two parallel lines; two skew lines. Are two vertical lines always parallel? Are two horizontal lines necessarily parallel? Illustrate.

6. A Line and a Plane. — If we think of the relative positions which a line and a plane may have, we find that the line may lie in the plane, may intersect the plane, or may not meet the plane.

Using your pencil to represent a line and your book cover to represent a plane, illustrate each of these possibilities. Make a drawing to show each. If a line has one point in a plane and is parallel to a line in the plane, where is the line located? If a line is parallel to a line in a plane and has no point in the plane, describe its position. Represent a line (the pencil) meeting a plane (the book cover) slantwise. Draw a line in the plane perpendicular to the given line at its foot. Can you draw in the plane more than one line perpendicular to the given line? Can we have a line meeting a plane so that the line will be perpendicular to all lines drawn in the plane passing through its foot? Describe the position of such a line.

7. Two Planes in Space. — Two planes may, or may not, have points in common. In what other way might we describe these two possibilities?

Can two distinct planes have only one common point? If two planes intersect and a straight line is drawn joining two points common to the two planes, where will the line lie? Could any point outside this line be common to both planes? How would you define the *intersection* of two planes?



8. Three Planes in Space. — Three planes may be so placed that they intersect in one line, in two lines, or in three lines; or they may not intersect at all.

The leaves of a book may be used to illustrate three planes meeting in one line. The ceiling, a side wall, and the floor represent three planes having two lines of intersection. Would two of the planes necessarily be parallel? What relation exists between the two lines of intersection? If three planes have three lines of intersection, are the lines of intersection necessarily parallel? Could the planes be placed so that the three lines all pass through the same point? Can three planes intersect in three lines only two of which are parallel? Can you find illustrations to justify your conclusions?

9. Definitions. — We now understand certain relations which we can state as definitions.

(a) The *foot* of a line which meets a plane is its point of contact with the plane.

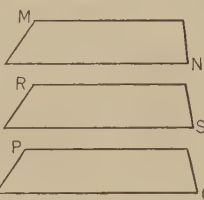
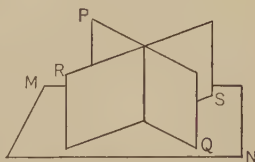
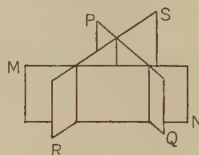
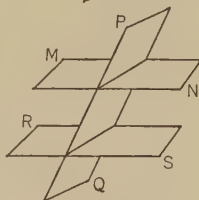
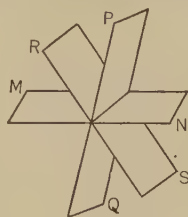
(b) The *intersection* of two surfaces is the line that contains all points common to the two surfaces.

(c) A straight *line perpendicular to a plane* is a line perpendicular to every line in the plane passing through its foot.

(d) A *line and a plane are parallel*, if they cannot meet however far produced.

(e) *Two planes are parallel*, if they cannot meet however far produced.

It follows from these definitions that a line, or a plane, not

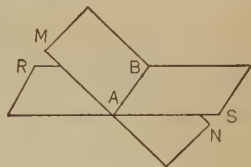


parallel to a given plane will meet it, if produced. Therefore we have the following :

10. Theorem. — If two planes meet, or intersect, their intersection is a straight line.

Given: The intersecting planes MN and RS .

To prove: MN and RS intersect in a straight line.



Proof: Let A and B be two points common to the planes MN and RS . Draw the straight line AB . All points in the line AB lie in the planes MN and RS (§ 4). No point not in AB can be common to MN and RS (§ 4). $\therefore$ the straight line AB is the intersection of MN and RS (§ 9b).

Note: In plane geometry a straight line is determined by two points. In solid geometry a straight line is also determined by two intersecting planes.

11. Summary. — In this introduction we have studied certain relations existing between lines and planes in space. We have seen that solid geometry differs from plane geometry by treating of figures not all parts of which lie in the same plane. The definitions and conclusions reached lead to the following assumptions :

(a) Through a straight line any number of planes may be passed.

(b) If two points of a straight line lie in a plane, the whole line lies in the plane.

(c) A straight line can intersect, or meet a plane in only one point.

(d) A plane is determined by a straight line and a point not in the line; by three points not in the same straight line; by two intersecting lines; or by two parallel lines.

The conclusions which we have drawn informally in this introduction we now shall prove in a formal way.

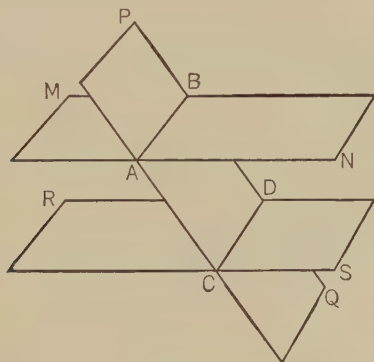
In Book I we are to study theorems based upon these relations between lines and planes in space.

BOOK ONE

LINES AND PLANES

Proposition I. Theorem

12. *If two parallel planes are cut by a third plane, the lines of intersection are parallel.*



Given: Plane $MN \parallel$ plane RS . Plane PQ cutting MN and RS in the lines AB and CD , respectively.

To prove: $AB \parallel CD$.

Plan of Attack: class — parallel lines.

method to be used — show that they are in the same plane and cannot meet.

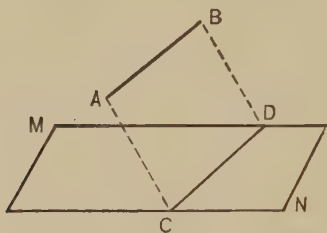
Proof:

- | | |
|---|-----------|
| 1. AB and CD are in the same plane PQ . | 1. § 9b. |
| 2. AB (in plane MN) cannot meet CD (in plane RS). | 2. § 9e. |
| 3. $\therefore AB \parallel CD$. | 3. § 388. |

13. Corollary. — Parallel lines included between parallel planes are equal.

Proposition II. Theorem

14. If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other.



Given: Line $AB \parallel$ line CD . Plane MN passing through CD but not through AB .

To prove: $AB \parallel MN$.

Plan of Attack: class — line $\parallel$ plane.

method to be used — show that they cannot meet.

Proof:

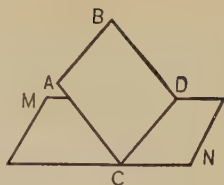
- | | |
|--|-----------|
| 1. Through AB and CD pass a plane AD . | 1. § 4. |
| 2. Then CD is the line of intersection of the planes MN and AD . | 2. § 9b. |
| 3. If AB meets MN , it must meet it in the line CD . | 3. § 9b. |
| 4. But the parallel lines AB and CD cannot meet. | 4. § 388. |
| 5. $\therefore AB \parallel MN$. | 5. § 9d. |

15. Method of Proof. — To prove that a line is parallel to a plane show that it is parallel to a line in the plane.

16. Corollary I. — If a line is parallel to a plane, the intersection of the plane with a plane passed through the line is parallel to the line.

Given: $AB \parallel \text{plane } MN$. Plane AD through AB intersects MN in CD .

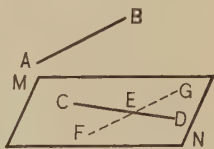
Prove: $AB \parallel CD$. (Show that AB and CD are in the same plane and that AB cannot meet CD which is in MN) (§ 9b, d).



17. Corollary II. — If a line and a plane are parallel, a parallel to the line through any point in the plane lies in the plane.

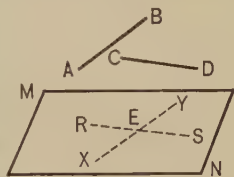
18. Corollary III. — Through either of two skew lines one plane, and only one, can be passed parallel to the other line.

Let AB and CD be two skew lines. Through point E in CD draw $FG \parallel AB$. Then $AB \parallel \text{plane } MN$ determined by CD and FG (§ 14). Every plane through $CD \parallel AB$ must contain FG (§ 17). Hence it must coincide with MN (§ 4).



19. Corollary IV. — Through a given point in space one plane, and only one, can be passed parallel to each of two skew lines, or else parallel to one line and containing the other.

Let AB and CD be two skew lines, and E any point in space. Through E draw $XY \parallel AB$ and $RS \parallel CD$. The plane MN determined by XY and RS is the required plane (§ 14). Any other plane through E parallel to AB and CD must contain XY and RS (§ 17) and hence coincide with MN (§ 4).

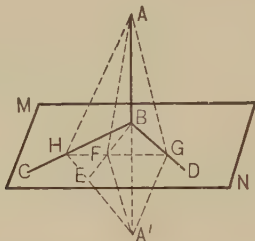


Exercise

Explain how to construct a line through a given point parallel to a given plane. Is only one line possible?

Proposition III. Theorem

20. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.



Given: $AB \perp BC$ and $AB \perp BD$. BC and BD intersecting in B . Plane MN determined by BC and BD .

To prove: $AB \perp MN$.

Plan of Attack: class — line $\perp$ plane.

method to be used — show that AB is $\perp$ to any other line in MN passing through B .

Analysis: $AB \perp BE$, any other line in MN through B , if points B and F are each equidistant from A and A' . $FA = FA'$, if $\triangle AGF = \triangle A'GF$; $\triangle AGF = \triangle A'GF$, if $\angle AGF = \angle A'GF$; $\angle AGF = \angle A'GF$, if $\triangle AGH = \triangle A'GH$; etc. (See Suggestion, page 157.)

Proof:

- | | |
|---|-----------------|
| 1. Through B in MN draw any other line BE . Through point F in BE draw GH cutting BD at G and BC at H . | 1. § 307. |
| 2. Extend AB to A' making $BA' = AB$. Draw AG , AF , $A'H$, $A'G$, $A'F$, and $A'H$. | 2. §§ 307; 309. |
| 3. Then BD and BC are $\perp$ bisectors of AA' . | 3. Why? |

- | | |
|---|---------------------|
| 4. In $\triangle AGH$ and $\triangle A'GH$, $GH = ?$,
$AG = ?$, $AH = ?$ | 4. Identity; § 324. |
| 5. $\therefore \triangle AGH = \triangle A'GH$, and $\angle AGH = \angle A'GH$ | 5. §§ 320c; 322. |
| 6. In $\triangle AGF$ and $\triangle A'GF$, $AG = ?$,
$GF = ?$, $\angle AGH = ?$ | 6. Why? |
| 7. $\therefore \triangle AGF = \triangle A'GF$ | 7. Why? |
| 8. $\therefore AF = A'F$ and point F is equi-
distant from A and A' . | 8. § 322. |
| 9. But point B is equidistant from
A and A' . | 9. Why? |
| 10. $\therefore BE \perp AB$, or $AB \perp BE$, any
other line in MN through B . | 10. § 323. |
| 11. $\therefore AB \perp MN$. | 11. § 9c. |

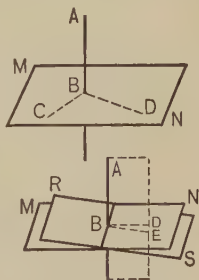
Questions: Why do we prove $\triangle AGH = \triangle A'GH$? $\angle AGH = \angle A'GH$? $\triangle AGF = \triangle A'GF$? $AF = A'F$? $AB \perp BE$?

21. Method of Proof. — To prove that a line is perpendicular to a plane, show that it is perpendicular to each of two intersecting lines in the plane.

22. Corollary I. — Through a given point in a given line one plane, and only one, can be passed perpendicular to the line.

Let AB be the given line and B a point in AB . Draw BC and BD , any two lines perpendicular to AB at B . Then MN , the plane determined by BC and BD , is the required plane (§ 20).

Suppose plane RS also $\perp AB$ at B . Through AB pass a plane AD intersecting RS in BE and MN in BD . Then $AB \perp BE$ and also $AB \perp BD$ (§ 9c). But this is impossible (§ 314). $\therefore MN$ is the only plane $\perp AB$ at B .

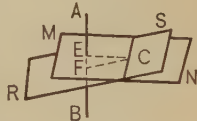
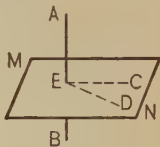


Question: Why can both BC and BD be $\perp AB$ at B ?

23. Corollary II. — Through a given external point one plane, and only one, can be passed perpendicular to a given line.

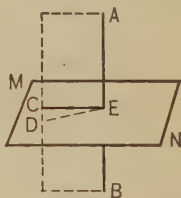
Let AB be the given line and C the given external point. Draw $CE \perp AB$ from point C . At E in the line AB draw $ED \perp AB$. Then MN , the plane determined by CE and ED , is the required plane (§ 20).

Suppose plane RS also drawn through $C \perp AB$. Through AB and C pass a plane AC cutting RS in FC and MN in CE . Then $AB \perp EC$ and also $AB \perp FC$ (§ 9c). But this is impossible (§ 314). $\therefore MN$ is the only plane through $C \perp AB$.



24. Corollary III. — All the perpendiculars that can be drawn to a given line at a given point lie in a plane which is perpendicular to the given line at the given point.

Let plane MN be perpendicular to line AB at point E and EC be any line $\perp AB$ at E . Draw the plane AC determined by AB and EC . This plane intersects MN in some line DE (§ 10). Then $AB \perp DE$ at point E (§ 9c). But $AB \perp EC$ at point E . $\therefore EC$ coincides with ED (§ 314). $\therefore EC$, any line $\perp AB$ at E , lies in MN .

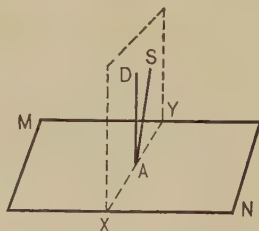
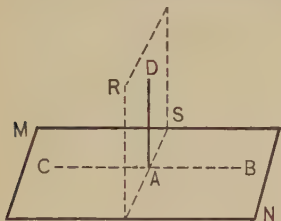


Exercises

1. What is the locus of all straight lines which make a right angle with the line AB at B ?
2. How many points are necessary to determine a plane
(a) parallel to a given line, (b) perpendicular to a given line?
3. If a pencil is held parallel to a wall, why is the shadow of the pencil on the wall parallel to the pencil?

Proposition IV. Theorem

25. *Through a given point in a plane there can be drawn one line, and only one, perpendicular to the plane.*



Given: Point A in the plane MN .

To prove: That through A there can be drawn one, and only one, line perpendicular to MN .

Plan of Attack: class — line $\perp$ plane.

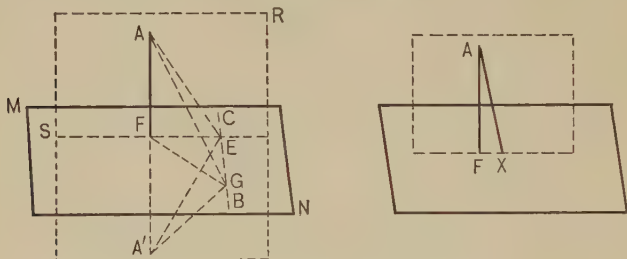
method to be used — a line $\perp$ to each of two intersecting lines at their point of intersection is $\perp$ plane determined by them.

Proof:

- | | |
|--|------------|
| 1. In plane MN draw through A the line BC . | 1. § 307. |
| 2. At A in line BC draw plane $RS \perp BC$. | 2. § 22. |
| 3. RS intersects MN in the line AS . | 3. Why? |
| 4. In RS draw $DA \perp SA$ at A . | 4. § 314. |
| 5. Then $BC \perp DA$, or $DA \perp BC$. | 5. § 9c. |
| 6. $\therefore DA \perp MN$. | 6. Why? |
| 7. If any other line AS could be drawn $\perp MN$ at A , then AS and AD would determine a plane DY . | 7. § 4. |
| 8. This plane DY would intersect MN in some line XY . | 8. Why? |
| 9. Then $AD \perp XY$ and $AS \perp XY$. | 9. § 9c. |
| 10. Since this is impossible, AD is the only line through $A \perp MN$. | 10. § 314. |

Proposition V. Theorem

26. *Through a given external point there can be drawn one line, and only one, perpendicular to a plane.*



Given: Plane MN and the external point A .

To prove: That through A there can be drawn one, and only one, line $\perp MN$.

Plan of Attack: class — line $\perp$ plane.

method to be used — a line perpendicular to each of two lines at their point of intersection is perpendicular to the plane determined by them.

Proof:

- | | |
|---|-----------------|
| 1. In plane MN draw a line BC ; pass a plane RS through point $A \perp BC$. | 1. § 23. |
| 2. This plane RS intersects MN in a straight line SE . | 2. Why? |
| 3. In RS draw $AF \perp SE$. | 3. § 314. |
| 4. In MN draw line FG from F to BC ; extend AF to A' making $AF = A'F$; draw $AE, AG, A'E$, and $A'G$. | 4. §§ 307; 309. |
| 5. $BC \perp AE$ and $BC \perp A'E$. | 5. § 9c. |
| 6. $\therefore \angle GEA$ and GEA' are right $\angle$. | 6. Why? |

7. In $\triangle GEA$ and GEA' , $GE = ?$, $AE = ?$, and $\angle GEA = \angle ?$	7. Identity; §§ 324; 318.
8. $\therefore \triangle GEA = \triangle GEA'$.	8. Why?
9. $\therefore AG = A'G$ and G is equi- distant from A and A' .	9. Why?
10. F is equidistant from A and A' .	10. Why?
11. $\therefore FG \perp AF$, or $AF \perp FG$.	11. § 323.
12. But $AF \perp FE$.	12. Why?
13. $\therefore AF \perp MN$.	13. § 20.
14. If any other line AX could be drawn from $A \perp MN$, then AF and AX would determine a plane.	14. Why?
15. This plane would intersect MN in a straight line FX , and AF and AX would both be $\perp$ to the line FX .	15. §§ 10; 9c.
16. Since this is impossible, AF is the only perpendicular that can be drawn from A to MN .	16. § 314.

27. Corollary. — The perpendicular is the shortest line from a point to a plane.

Let AF be the $\perp$ and AX be any other line from point A to the plane MN . Through AF and AX pass a plane intersecting MN in line FX . Then $AF \perp FX$ (Why?), and $AF < AX$ (§ 375).

28. Definition. The *distance from a point to a plane* is the length of the perpendicular from the point to the plane.

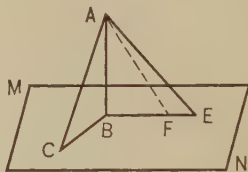
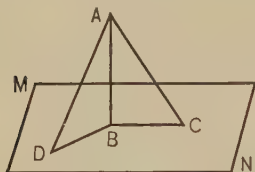
Exercise

If three planes, not passing through the same line, intersect each other, their three lines of intersection pass through the same point, or else they are parallel each to each. (See page 158.)

Proposition VI. Theorem

29. I. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and*

II. *Of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the one corresponding to the greater distance is the greater.*



Given: I. $AB \perp$ plane MN at B . AD and AC oblique lines drawn from A to plane MN so that $BC = BD$.

To prove: $AC = AD$.

Plan of Attack: class — equal lines.

method to be used — corresponding parts of congruent $\triangle$ are equal.

Proof:

- | | |
|--|-------------------------------------|
| 1. $AB \perp MN$. | 1. Why? |
| 2. $\therefore AB \perp BC$ and $AB \perp BD$. | 2. § 9c. |
| 3. In the $\triangle ABC$ and $\triangle ABD$, $AB = ?$,
$BC = ?$, $\angle ABC = \angle ?$ | 3. Identity; given;
§§ 386; 318. |
| 4. $\therefore \triangle ABC = \triangle ABD$. | 4. Why? |
| 5. $\therefore AC = AD$. | 5. Why? |

Given: II. $AB \perp$ plane MN at B . AE and AC oblique lines from A to plane MN ; $BE > BC$.

To prove: $AE > AC$.

Plan of Attack: class — unequal lines.

method to be used — § 377, and Ax. VIII.

Proof:

- | | |
|--|------------------|
| 1. AB and AE determine a plane. | 1. Why? |
| 2. On BE lay off $BF = BC$, and draw AF . | 2. §§ 307 ; 309. |
| 3. $BE > BF$. | 3. Why? |
| 4. $\therefore AE > AF$. | 4. § 377. |
| 5. But $AC = AF$. | 5. § 29, I. |
| 6. $\therefore AE > AC$. | 6. Why? |

30. Corollary. — Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular drawn from the point to the plane.

31. Corollary. — The locus of points equidistant from the vertices of a triangle is the line through the center of the circumscribed circle perpendicular to the plane of the triangle.

Exercises

1. If two planes intersect, a straight line drawn parallel to the intersection through any point not in the planes is parallel to each of the planes.

2. Parallel lines drawn to a plane from points in a line parallel to the plane are equal.

3. If a line in one of two intersecting planes is parallel to a line in the other plane, the intersection of the planes is parallel to each of the given parallel lines.

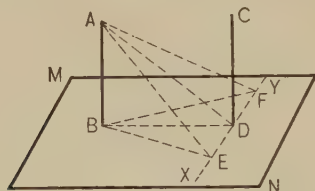
4. If a plane, parallel to the intersection of two intersecting planes, intersects each of the planes, the lines of intersection are parallel.

5. If a line is parallel to each of two intersecting planes, it is parallel to their intersection.

6. If from the foot of a perpendicular to a plane a line is drawn making right angles with a line in the plane, the line drawn from the intersection of the two lines in the plane to any point in the perpendicular is perpendicular to the line in the plane.

Proposition VII. Theorem

32. Two lines perpendicular to the same plane are parallel.



Given: Plane MN , $AB \perp MN$ and $CD \perp MN$.

To prove: $AB \parallel CD$.

Plan of Attack: class — parallel lines.

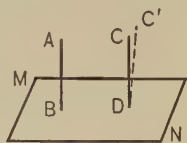
method to be used — show that they are in the same plane and $\perp$ to the same line.

Proof:

- | | |
|--|------------------------|
| 1. Draw BD and AD . In the plane MN draw $XY \perp BD$; on XY mark $DE = DF$. Draw AE , AF , BE , and BF . | 1. §§ 307; 314. |
| 2. Then $BE = ?$ | 2. § 324. |
| 3. But $AB \perp MN$. | 3. Why? |
| 4. $\therefore AE = ?$ and point A is equidistant from points E and F . | 4. § 29. |
| 5. But the point D is equidistant from points E and F . | 5. Why? |
| 6. $\therefore AD \perp EF$, or XY . | 6. § 323. |
| 7. But $BD \perp XY$ and $CD \perp XY$. | 7. Construction; § 9c. |
| 8. $\therefore BD$, AD , and CD are in the same plane. | 8. § 24. |
| 9. $\therefore AB$ also lies in this plane. | 9. § 11b. |
| 10. $AB \perp BD$ and $CD \perp BD$. | 10. Why? |
| 11. $\therefore AB \parallel CD$. | 11. § 336. |

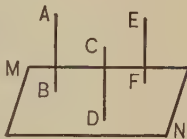
33. Corollary I. — If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.

Given $AB \parallel CD$ and $AB \perp MN$. Prove $CD \perp MN$. Draw $C'D \perp MN$ at D (§ 25). Then $C'D \parallel AB$ (§ 32). $\therefore CD$ coincides with $C'D$ (§ 313). $\therefore CD \perp MN$. Why?



34. Corollary II. — If two lines are parallel to a third line, they are parallel to each other.

Given $AB \parallel CD$ and $EF \parallel CD$. Prove $AB \parallel EF$. Draw $MN \perp CD$ (§ 22). Then $AB \perp MN$ and $EF \perp MN$ (§ 33). $\therefore AB \parallel EF$ (§ 32).



35. Corollary III. — Two parallel planes are everywhere equidistant.

36. The *distance between two parallel planes* is the perpendicular distance from any point in one plane to the other plane.

Exercises

1. Lines joining the midpoints of the adjacent sides of a quadrilateral in space form a parallelogram. (A quadrilateral in space has two adjacent sides in one plane and the other two sides in another plane.)

2. The line joining the midpoints of two opposite sides of a quadrilateral in space bisects the line joining the midpoints of the other two sides.

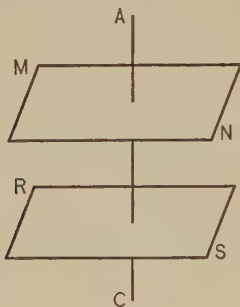
3. If a plane is passed through one diagonal of a parallelogram, it is equally distant from the ends of the other diagonal.

4. Determine a point in a plane which shall be equidistant from three given points in space.

5. Why can we not prove § 34 by the same method that was used in plane geometry?

Proposition VIII. Theorem

37. *Two planes perpendicular to the same line are parallel.*



Given: Line $AC \perp$ the plane MN and $AC \perp$ the plane RS .

To prove: $MN \parallel RS$.

Plan of Attack: class — parallel planes.

method to be used — show that they cannot meet.

Proof:

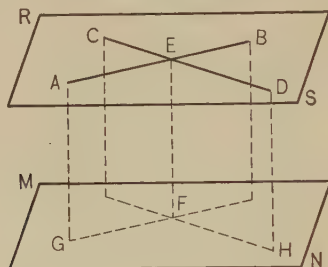
- | | |
|--|----------|
| 1. If MN and RS are not parallel, they must meet at some point. | 1. § 9e. |
| 2. But this is impossible, for MN and RS would then be two planes through a point perpendicular to the same straight line. | 2. § 23. |
| 3. $\therefore MN \parallel RS$. | 3. § 9e. |

Exercises

1. If two lines are parallel to the same plane, are the lines necessarily parallel?
2. Prove that two parallel planes are everywhere equidistant.
3. A parallel to a plane is everywhere equidistant from it.
4. If one of two parallel lines is parallel to a plane, the other, if outside the plane, is parallel to the plane.

Proposition X. Theorem

40. If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.



Given: The line AB intersecting the line CD in E , and the plane RS determined by them; $AB \parallel$ the plane MN and $CD \parallel MN$.

To prove: $RS \parallel MN$.

Plan of Attack: class — parallel planes.

method to be used — two planes $\perp$ the same line are parallel.

Proof:

- | | |
|--|--------------|
| 1. Through E draw $EF \perp MN$. | 1. § 26. |
| 2. Through AB and EF pass a plane intersecting MN in GF ; through CD and EF pass a plane intersecting MN in FH . | 2. §§ 4; 10. |
| 3. Then $AB \parallel ?$ and $CD \parallel ?$ | 3. § 16. |
| 4. But $EF \perp GF$ and FH . | 4. Why? |
| 5. $\therefore EF \perp AB$ and CD . | 5. § 338. |
| 6. $\therefore EF \perp RS$. | 6. Why? |
| 7. $\therefore RS \parallel MN$. | 7. § 37. |

41. **Corollary.** — If two intersecting lines in one plane are parallel respectively to two intersecting lines in another plane, the plane of the first pair is parallel to the plane of the second pair.

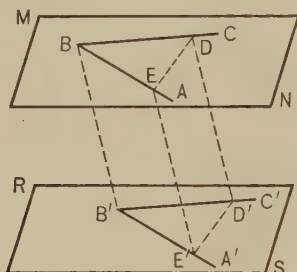
(Use §§ 14 and 40.)

Questions

1. Why is a fold in a sheet of paper a straight line?
2. Is it possible to connect two points in the surface of a ball so that the connecting line is a straight line?
3. Can a straight line be drawn connecting two points on the surface of a stove pipe? Is the stove pipe a plane surface?
4. How many planes are determined by three lines passing through the same point, but not all in the same plane?
5. Why is the transversal of two parallel lines in the plane of parallels?
6. Which of the following are necessarily plane figures: a triangle; a quadrilateral; a parallelogram; a trapezoid?
7. Is it possible for two curved surfaces to intersect in a straight line?
8. If you are asked to prove two planes parallel, what theorems would you try to use?
9. In our work in solid geometry what lines have been proved parallel?
10. Are all lines, which are perpendicular to a vertical line, horizontal? Are all lines, which are perpendicular to a horizontal line, vertical?
11. Can two skew lines be vertical? horizontal?
12. Can one of two skew lines be in one plane and the other in a parallel plane?
13. Two intersecting lines determine a plane. Do two intersecting planes determine a line?
14. Two lines perpendicular to a plane are parallel? Are two planes perpendicular to a line parallel?
15. Two lines parallel to the same line are parallel. Are two planes parallel to the same plane parallel?
16. If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other. If two planes are parallel, is every line in one of the planes parallel to the other?

Proposition XI. Theorem

42. If two angles not in the same plane have their sides respectively parallel and in the same sense, they are equal and their planes are parallel.



Given: $\angle ABC$ and $A'B'C'$, with $AB \parallel A'B'$ and $BC \parallel B'C'$ and in the same sense; planes MN and RS determined by $\angle ABC$ and $A'B'C'$, respectively.

To prove: $\angle ABC = \angle A'B'C'$ and $MN \parallel RS$.

Plan of Attack: class — equal $\angle$; parallel planes.

methods to be used — corresponding parts of congruent triangles are equal; and § 41.

Proof:

- | | |
|--|-----------------|
| 1. On the sides lay off $BD = B'D'$ and $BE = B'E'$. Draw BB' , DD' , EE' , DE , and $D'E'$. | 1. §§ 308; 309. |
| 2. $BD =$ and $\parallel$? | 2. Why? |
| 3. $\therefore BB'D'D$ is a $\square$. | 3. § 339c. |
| 4. $\therefore BB' =$ and $\parallel$? | 4. Why? |
| 5. In the same manner, $BB' =$ and $\parallel$? | 5. Reasons 2-4. |
| 6. $\therefore DD' =$? | 6. § 317, VII. |
| 7. and $DD' \parallel$? | 7. § 34. |
| 8. $\therefore EE'D'D$ is a $\square$. | 8. § 339c. |
| 9. $\therefore DE =$? | 9. Why? |
| 10. $\therefore \triangle BED = \triangle B'E'D'$. | 10. Why? |

- | | |
|---|------------|
| 11. $\therefore \angle ABC = \angle A'B'C'$. | 11. Why? |
| 12. $BC \parallel ?$ and $AB \parallel ?$ | |
| 13. $\therefore MN \parallel RS$. | |
| | 12. Given. |
| | 13. § 41. |

43. Method of Proof. — To prove two angles are equal, show that their sides are respectively parallel and in the same sense.

Exercises

1. If three planes are parallel to the same line, are they necessarily parallel to each other?

2. Given two parallel planes and a line intersecting both of them. Through the line two planes are drawn intersecting the parallel planes. Prove that their intersections with the parallel planes include equal angles.

3. Two planes parallel to the same plane are parallel.

4. If from any point in a perpendicular to a plane a line is drawn making right angles with a line in the plane, the line from its intersection with the line in the plane to the foot of the perpendicular to the plane is perpendicular to the line in the plane.

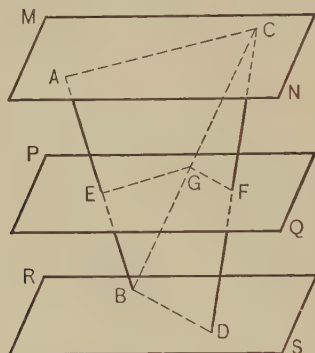
5. Given the parallel planes MN , PQ , and RS ; PQ lying between MN and RS . A is any point in MN and B and C are any two points in RS . Lines AB and AC intersect PQ in the points D and E , respectively. Prove $\frac{AD}{DB} = \frac{AE}{EC}$.

6. Given the parallel planes MN , PQ , and RS . PQ lies between MN and RS . From the points A and C in MN the lines AB and CD are drawn through point E in PQ to the points B and D in plane RS . Prove $\frac{AE}{EB} = \frac{CE}{ED}$.

7. The parallel lines AB and CD are cut by the parallel planes MN , PQ , and RS in the points A , E , B and C , F , D , respectively. Prove $\frac{AE}{EB} = \frac{CF}{FD}$.

Proposition XII. Theorem

44. If two lines are cut by three parallel planes, their corresponding segments are proportional.



Given: Lines AB and CD , cut by the parallel planes MN , PQ , and RS in the points A , E , B and C , F , D , respectively.

To prove: $\frac{AE}{EB} = \frac{CF}{FD}$.

Plan of Attack: class — proportional lines.

method to be used — a line parallel to one side of a triangle divides the other two sides proportionally; and Axiom VII.

Proof:

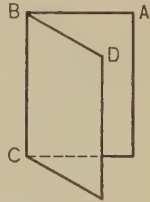
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|---|----------------|
| 1. Draw CB cutting the plane PQ in G . | 1. § 307. |
| 2. Pass planes through AB and BC , and through BC and CD intersecting the planes in the lines AC and EG , and GF and BD , respectively. | 2. §§ 4; 10. |
| 3. Then $AC \parallel ?$ and $GF \parallel ?$ | 3. § 12. |
| 4. $\therefore \frac{AE}{EB} = ?$ and $\frac{CF}{FD} = ?$ | 4. § 334. |
| 5. $\therefore \frac{AE}{EB} = \frac{CF}{FD}$. | 5. § 317, VII. |

Dihedral Angles

45. A *dihedral angle* is the figure formed by two planes which meet.

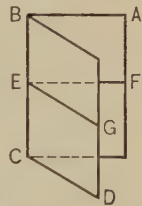
The *faces* of a dihedral angle are the two planes forming the dihedral angle. The *edge* of the dihedral angle is the line of intersection of the faces.

A dihedral angle is indicated by naming a point in one face, the edge, and a point in the other face; as, the dihedral $\angle A-BC-D$. When there is no doubt as to the meaning, a dihedral angle may be indicated by naming its edge; as, the dihedral $\angle BC$.



46. The *plane angle* of a dihedral angle is the angle formed by two lines, one in each face, each perpendicular to the edge at the same point.

In the dihedral angle $A-BC-D$, EF in the face AC is perpendicular to BC at E and EG in the face BD is perpendicular to BC at E ; hence, $\angle FEG$ is the plane angle of the dihedral angle.



47. A *right dihedral angle* is a dihedral angle whose plane angle is a right angle.

48. Two *planes* are *perpendicular to each other*, if they form a right dihedral angle.

49. Method of Proof. — To prove two planes are perpendicular to each other, show that they form a dihedral angle whose plane angle is a right angle.

50. Related Pairs of Dihedral Angles. — Following the analogy of the corresponding definitions of plane geometry, we define adjacent, vertical, supplementary, and complementary dihedral angles. Thus, adjacent dihedral angles

are dihedral angles that have a common edge and a common face between them. You will notice that the plane geometry definition is changed by substituting a plane for a line, and a line for a point.

Theorems Proved Informally

51. All plane angles of the same dihedral angle are equal.

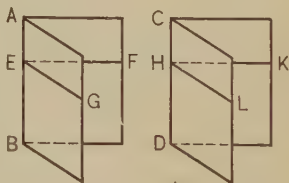
In the figure, if $\angle EFG$ and $\angle HKL$ are plane $\angle$ s of the dihedral $\angle A-BC-D$, $EF \parallel KH$ and $FG \parallel KL$ (§ 336). $\therefore \angle EFG = \angle HKL$ (§ 42).



52. Two dihedral angles are equal, if their plane angles are equal.

Given $\angle FEG$, the plane angle of the dihedral $\angle F-AB-G$, equal to $\angle KHL$, the plane angle of the dihedral $\angle K-CD-L$.

$\angle FEG$ can be made to coincide with $\angle KHL$. Why? $\therefore$ the plane FEG coincides with the plane KHL . Why? $AB \perp$ plane FEG and $CD \perp$ plane KHL . Why? $\therefore AB$ coincides with CD (§ 25). $\therefore$ the face AF coincides with face CK and face AG with CL (§ 4). $\therefore$ the dihedral $\angle F-AB-G$ coincides with the dihedral $\angle K-CD-L$.



53. If two dihedral angles are equal, their plane angles are equal.

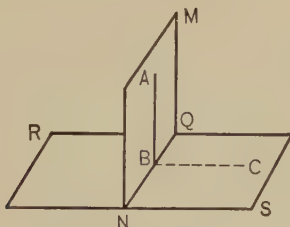
54. Two dihedral angles have the same ratio as their plane angles.

55. A dihedral angle is measured by its plane angle.

56. Vertical dihedral angles are equal.

Proposition XIII. Theorem

57. *If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.*



Given: The line $AB \perp$ the plane RS at B ; MN , any plane containing AB and intersecting RS in NQ .

To prove: $MN \perp RS$.

Plan of Attack: class — perpendicular planes.

method to be used — two planes are perpendicular, if they form a right dihedral $\angle$.
(Show that the plane $\angle$ of the dihedral is a right $\angle$.)

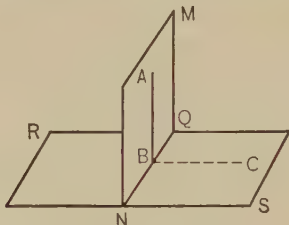
Proof:

- | | |
|--|-----------|
| 1. In RS draw $BC \perp NQ$ at B . | 1. § 314. |
| 2. $AB \perp NQ$ at B . | 2. § 9c. |
| 3. $\therefore \angle ?$ is the plane $\angle$ of the dihedral $\angle M-NQ-S$. | 3. § 46. |
| 4. $AB \perp BC$. | 4. § 9c. |
| 5. $\therefore \angle ?$ is a right $\angle$. | 5. Why? |
| 6. $\therefore$ the dihedral $\angle M-NQ-S$ is a right dihedral angle. | 6. § 47. |
| 7. $\therefore MN \perp RS$. | 7. § 48. |

58. Method of Proof. — To prove that two planes are perpendicular show that one plane contains a line that is perpendicular to the other plane.

Proposition XIV. Theorem

59. *If two planes are perpendicular, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given: Plane $MN \perp$ plane RS ; NQ their line of intersection; AB in $MN \perp NQ$ at the point B .

To prove: $AB \perp RS$.

Plan of Attack: class — line $\perp$ plane.

method to be used — prove AB perpendicular to each of two intersecting lines in RS .

Proof:

- | | |
|---|-----------|
| 1. In RS draw $BC \perp NQ$ at B . | 1. § 314. |
| 2. Then $\angle ?$ is the plane angle of the dihedral $\angle M-NQ-S$. | 2. § 46. |
| 3. $MN \perp RS$. | 3. Why? |
| 4. $\therefore \angle M-NQ-S$ is a right dihedral angle. | 4. § 48. |
| 5. $\therefore \angle ABC$ is a right angle. | 5. § 47. |
| 6. $\therefore AB \perp ?$ | 6. Why? |
| 7. But $AB \perp ?$ | 7. Given. |
| 8. $\therefore AB \perp RS$. | 8. Why? |

Note: This theorem furnishes a new method of proving a line perpendicular to a plane.

60. Corollary I. — If two planes are perpendicular, a line perpendicular to one of them at any point of their intersection will lie in the other.

In the figure for § 59, $MN \perp RS$ and $AB \perp RS$ at the point B in the line of intersection NQ . Prove AB lies in MN .

In MN draw $A'B \perp NQ$ at B . Then $A'B \perp RS$. Why?
 $\therefore AB$ coincides with $A'B$ (§ 25). $\therefore AB$ lies in MN . Why?

61. Corollary II. — If two planes are perpendicular, a line drawn perpendicular to one of them through any point in the other will lie in the second plane.

Exercises

1. State a proposition that may be used to construct (a) a plane parallel to a given line, (b) a plane perpendicular to a given line.

2. Show how to construct a plane through a given point, parallel to a given line and perpendicular to a given plane.

3. A plane perpendicular to the edge of a dihedral angle is perpendicular to the faces.

4. If a plane is perpendicular to a line in another plane, are the planes necessarily perpendicular to each other?

5. If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular.

6. Are two planes both of which are perpendicular to the same plane necessarily parallel?

7. If a plane parallel to the edge of a dihedral angle intersects the faces of the angle, its intersections with the faces are parallel lines.

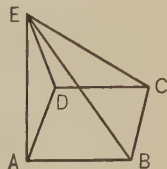
8. In the figure, $ABCD$ is a square; $EA \perp$ plane AC .

(a) Which planes are perpendicular? Why?

(b) $\triangle EBC$ and EDC are what kind of triangles? Why?

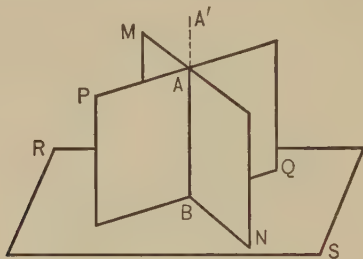
(c) Name the plane angles of the dihedral $\angle A-BC-E$ and $A-DC-E$.

(d) If $EA = AB$, how large are the dihedral $\angle A-BC-E$ and $A-DC-E$?



Proposition XV. Theorem

62. *If each of two intersecting planes is perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Given: Planes MN and PQ intersecting in AB ; $MN \perp RS$ and $PQ \perp RS$.

To prove: $AB \perp RS$.

Plan of Attack: class — line $\perp$ plane.

method to be used — show that AB coincides with a line that is perpendicular to RS .

Proof:

- | | |
|---|----------|
| 1. Through the point B , common to the three planes, erect $BA' \perp RS$. | 1. § 25. |
| 2. Then BA' lies in plane MN and also in plane RS . | 2. § 60. |
| 3. $\therefore BA'$ coincides with AB , the intersection of MN and RS . | 3. § 9b. |
| 4. $\therefore AB \perp RS$. | 4. Why? |

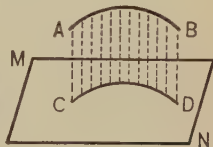
Note: This theorem furnishes another method of proving a line perpendicular to a plane.

63. Corollary I. — Through a given line, not perpendicular to a given plane, one plane, and only one, can be passed perpendicular to the given plane.

Given line AB not perpendicular to plane MN . Through any point C in line AB drop a line $CD \perp MN$ (§ 26). Then the plane determined by AB and $CD \perp MN$ (§ 57). If more than one plane could be passed through $AB \perp MN$, AB would be $\perp MN$ (§ 62).

64. The **projection of a point** on a plane is the foot of the perpendicular drawn from the point to the plane.

65. The *projection of a line* on a plane is the locus of the projections of all the points of the line on the plane :



CD is the projection of AB on plane MN .

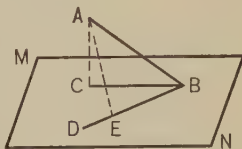
66. Corollary. — The projection of a straight line on a plane not perpendicular to it is a straight line.

Through the line pass a plane $\perp$ the given plane. Since this plane contains all the $\perp$ drawn to the given plane from points in the given line (§ 61), the projection must coincide with the intersection of the two planes (§ 9b). $\therefore$ the projection is a straight line (§ 10).

67. The *angle which a line makes with a plane* is the angle which it makes with its projection on the plane.

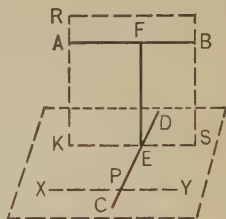
68. Theorem. — The acute angle formed by a straight line with its projection on a plane is the least angle which it makes with any line in that plane.

Outline of Proof: Let AB be the line, MN the plane, and BC the projection of AB on plane MN . Draw *any other* line BD in MN through B . Lay off on BD a distance $BE = BC$. Draw AE . Show that $AE > AC$ (§ 27), hence by § 376 $\angle ABD > \angle ABC$.



69. Theorem. — Between any two skew lines there is one common perpendicular, and only one.

Let AB and CD be two skew lines. Through P , any point in CD , draw $XY \parallel AB$. Draw the plane determined by CD and XY . Through AB pass a plane $RS \perp$ to this plane (§ 63) and meeting it in KS . At E , the intersection of CD and KS in the plane, erect $EF \perp$ to the plane (§ 25). Then $EF \perp CD$ (§ 9c) and KS . $\therefore EF \perp AB$ (§§ 14, 16, and 338), etc.



Exercises

1. Can the projection of a curve on a plane be a straight line?
2. Can the projection of a triangle upon a plane be a straight line?
3. A line parallel to a plane is parallel and equal to its projection on the plane.
4. If two equal lines are drawn to a plane from a point outside the plane, they make equal angles with the plane.
5. A line not parallel to a plane is longer than its projection on the plane.
6. If from any point within a dihedral angle perpendiculars are drawn to the faces, the plane determined by these perpendiculars is perpendicular to the edge of the dihedral angle.
7. If two lines, oblique to a plane, are parallel, their projections on the plane are either the same line or are parallel lines.
8. If two parallel lines are oblique to a plane, they make equal angles with the plane.
9. If the projections of a number of points on a given plane lie in a straight line, the points lie in a plane.
10. If two equal oblique lines are parallel and meet a plane, their projections on the plane are equal.

11. If two parallel line segments terminate in a plane, their projections upon the plane have the same ratio as the line segments themselves.

12. Given the plane MN and two oblique lines AB and CD not meeting MN . $AB = CD$ and $AB \parallel CD$. Prove that the projections of AB and CD on MN are equal.

13. If a line is perpendicular to one of two intersecting planes, its projection upon the other plane is perpendicular to the intersection of the two planes.

14. From any point within a dihedral angle perpendiculars are drawn to each of the faces. Prove that the angle formed by these lines is supplementary to the plane angle of the dihedral angle.

15. From any point in one of the faces of a dihedral angle perpendiculars are drawn to each of the faces. Prove that the angle formed by these lines is equal to the plane angle of the dihedral angle.

16. If three planes intersecting in three lines are all perpendicular to a fourth plane, their lines of intersection are parallel.

17. If a plane is perpendicular to one of two perpendicular planes, its intersection with the other plane is perpendicular to each of the other lines of intersection.

18. If a straight line intersects two parallel planes, it makes equal angles with them.

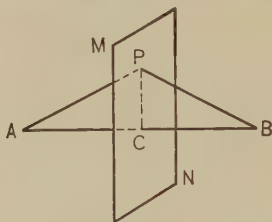
19. Is the projection of an angle on a plane necessarily equal to the angle? Is the projection of an acute angle on a plane necessarily an acute angle? Is the projection of a right angle on a plane necessarily a right angle?

20. If one side of a right angle lies in a plane, the projection of the angle on the plane is a right angle.

21. A square $ABCD$ has its side AB in the plane MN . The plane AC makes a 30° angle with plane MN . If the area of the square is 64 square inches, what is the area of the projection of $ABCD$ on plane MN ?

Proposition XVI. Theorem

70. *The locus of points equidistant from two given points is the plane perpendicular to the line joining them, at its midpoint.*



Given: Two points A and B , and C the midpoint of the line AB ; plane $MN \perp AB$ at C .

To prove: MN the locus of points equidistant from A and B .

Plan of Attack: class — locus of points.

method to be used — (a) prove that any point P in MN is equidistant from A and B , and (b) prove that any point P , equidistant from A and B , is in MN .

Proof:

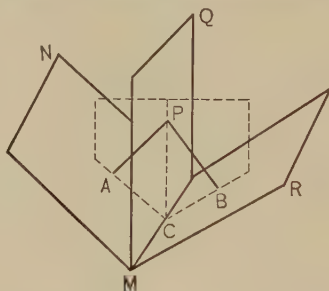
- | | |
|---|-----------------|
| 1. (a) Through P , any point in MN , draw AP , BP , and PC . | 1. § 307. |
| 2. $\triangle APC = \triangle BPC$. | 2. Give proof. |
| 3. $\therefore AP = PB$. | 3. Why? |
| (b) Let P be any point so that $PA = PB$. | |
| 4. In the plane determined by AP and PB , draw PC . | 4. § 307. |
| 5. $\triangle APC = \triangle BPC$. | 5. Give proof. |
| 6. $\therefore \angle ACP = \angle ?$ and $\angle ACP = \text{rt. } \angle$. | 6. §§ 322; 385. |
| 7. $\therefore PC \perp AB$ at C . | 7. Why? |
| 8. $\therefore PC$ lies in MN . | 8. § 24. |
| 9. $\therefore MN$ is the locus of points equidistant from A and B . | 9. Why? |

Exercises

1. What is the locus of points equidistant from two given parallel lines?
2. What is the locus of points a given distance from a given plane?
3. Given a plane and two points not in the plane, what is the locus of points that lie in the plane and that are equidistant from the two given points?
4. Find the point in a plane to which lines may be drawn from two given external points on the same side of the plane so that their sum shall be the least possible.
5. If the points in a line satisfy one condition and the points in a plane satisfy another condition, what will be true of their intersection?
6. If the points in one plane satisfy one condition and the points in another plane satisfy another condition, what will be true of their intersection?
7. What is the locus of points 6 inches from a given plane and equidistant from two given points?
8. Find the locus of points equidistant from two given points in space and also equidistant from two parallel planes.
9. Find a point equidistant from two given points, A and B , and also equidistant from two other points, C and D .
10. Find the locus of points equidistant from two given points in space and also equidistant from two parallel lines.
11. Find a point equidistant from four given points not all in the same plane.
12. Prove that the locus of points equidistant from the sides of an angle is the plane passing through the bisector of the angle perpendicular to the plane of the angle.
13. Find the locus of a point equidistant from three given points in space.

Proposition XVII. Theorem

71. *The locus of points equidistant from the faces of a dihedral angle is the plane bisecting the dihedral angle.*



Given: Plane MQ bisecting the dihedral angle formed by the planes MN and MR .

To prove: MQ is the locus of points equidistant from MN and MR .

Plan of Attack: class — locus of points.

method to be used — (a) prove that P , any point in MQ , is equidistant from MN and MR , and (b) prove that any point equidistant from MN and MR is in MQ .

Proof:

- | | |
|---|----------------|
| 1. (a) From P , any point in MQ , drop PA and $PB \perp MN$ and MR , respectively. | 1. § 26. |
| 2. Through PA and PB pass a plane intersecting MN in AC , MR in BC , and MQ in PC . | 2. §§ 4; 10. |
| 3. Then plane $APB \perp MN$ and MR . | 3. § 57. |
| 4. $\therefore MC \perp ?$, $?$, and $?$ | 4. §§ 62; 9c. |
| 5. $\therefore \angle ACP$ and $\angle BCP$ are the plane $\angle$ of the dihedral $\angle N-MC-Q$ and $R-MC-Q$. | 5. § 46. |
| 6. $\therefore \angle ACP = \angle BCP$ | 6. Why? |
| 7. Right $\triangle ACP =$ right $\triangle BCP$. | 7. Give proof. |

- | | |
|--|-----------------|
| 8. $\therefore PA = PB$. | 8. Why? |
| (b) Let P be any point so that
$\perp PA = \perp PB$. | |
| 9. Through PA and PB pass a plane intersecting MN in AC and MR in BC . Draw PC . | 9. §§ 4; 10. |
| 10. Through PC and MC pass a plane. | 10. § 4. |
| 11. $\angle$? and ? are the plane $\angle$ of the dihedral $\angle N-MC-P$ and $R-MC-P$. | 11. Give proof. |
| 12. Right $\triangle PAC =$ right $\triangle PBC$. | 12. Give proof. |
| 13. $\therefore \angle ACP = \angle ?$ | 13. § 322. |
| 14. $\therefore$ dihedral $\angle N-MC-P =$ dihedral $\angle R-MC-P$. | 14. § 52. |
| 15. $\therefore$ plane PCM bisects the dihedral angle $N-MC-R$ and coincides with plane MQ . | 15. Why? |
| 16. $\therefore MQ$ is the locus of points equidistant from MN and MR . | 16. Why? |

Exercises

- What is the locus of points (a) equidistant from two parallel planes; (b) equidistant from two intersecting planes; (c) equidistant from three points not in the same straight line; (d) equidistant from three planes meeting in a point; (e) equidistant from three non-parallel planes which are perpendicular to the same plane?
- Find the locus of points equidistant from two given points and also equidistant from two intersecting planes.
- Find a line whose points are equidistant from the faces of a dihedral angle and also equidistant from two points one in each face of the dihedral angle.
- The isosceles triangle ABC has its base BC in the plane MN . The plane of ABC makes an angle of 45° with the plane MN . If $BC = 6$ inches and its area is 24 square inches, what is the area of the projection of $\triangle ABC$ on plane MN ?

Polyhedral Angles

72. A **polyhedral angle** is the figure formed by concurrent lines of intersection of three or more planes, together with the included portions of the planes.

The *vertex* of the polyhedral angle is the point at which the planes forming the angle meet. The *edges* are the lines of intersection of the planes. The *faces* are the portions of the planes included between the edges. The *face angles* are the angles formed by the adjacent edges.

The parts of a polyhedral angle are its face angles and dihedral angles.

A polyhedral angle is indicated by naming its vertex together with a point in each edge; as, $P-ABCDE$; or by naming only its vertex, as polyhedral $\angle P$.



73. A **convex polyhedral angle** is a polyhedral angle in which a convex polygon is formed by the intersection of any plane with all the faces of the polyhedral angle.

Only convex polyhedral angles will be considered in this text.

74. A **trihedral angle** is a polyhedral angle having three faces. An *isosceles trihedral angle* has two of its face angles equal.



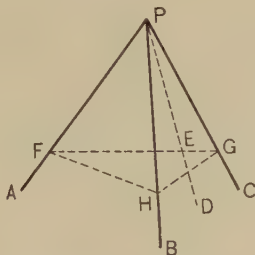
Exercises

1. The three planes bisecting the three dihedral angles of a trihedral angle intersect in a straight line, which is the locus of points equidistant from the faces of the trihedral angle.

2. The three planes passing through the three bisectors of the face angles of a trihedral angle perpendicular to those faces intersect in a straight line which is the locus of points equidistant from the edges of the trihedral angle.

Proposition XVIII. Theorem

75. The sum of any two face angles of a trihedral angle is greater than the third face angle.



Given: The trihedral angle $P-ABC$, with the face angles APB , BPC , and APC ; $\angle APC$ being the largest angle.

To prove: $\angle APB + \angle BPC > \angle APC$.

Plan of Attack: class — inequalities.

method to be used — show that $\angle APB = \angle APD$ and that $\angle BPC > \angle DPC$.

Proof:

1. In the face APC draw PD making $\angle APD = \angle APB$. Through E , any point in PD , draw the line FEG .

2. On PB lay off $PH = PE$, and pass a plane through FG and the point H , intersecting the other faces in FH and HG .

3. $\triangle FPH = \triangle FPE$. $\therefore FH = ?$

4. $FH + HG > FG$, or $FE + EG$.

5. $\therefore HG > EG$.

6. In $\triangle HPG$ and EPG , $PH = ?$, $PG = ?$, and $HG > ?$

7. $\therefore \angle HPG > \angle EPG$.

8. $\therefore \angle APB + \angle HPG > \angle APD + \angle EPG$.

9. Or $\angle APB + \angle BPC > \angle APC$.

1. § 316.

§ 307.

2. §§ 309; 4;
10.

3. Give proof.

4. § 310.

5. § 317, x.

6. Constr.;

identity; proved.

7. § 376.

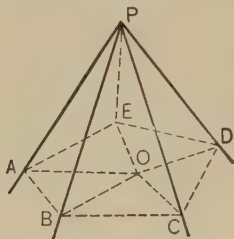
8. § 317, x.

9. §§ 317, via;

VIII.

Proposition XIX. Theorem

76. *The sum of all the face angles of any convex polyhedral angle is less than four right angles.*



Given: The polyhedral angle P .

To prove: The sum of the face angles at $P < 4$ right angles.

Plan of Attack: class — sum of angles.

method to be used — show that the sum of the face $\angle <$ the sum of the angles around a point in a plane.

Proof:

- | | |
|--|---------------|
| 1. Pass a plane cutting all the faces forming the polygon $ABCDE$. | 1. §§ 4; 10. |
| 2. Connect any point O within the polygon $ABCDE$ with the vertices A, B, C, D , and E . | 2. § 307. |
| 3. The sum of all the angles of the $\triangle$ having P as a vertex = the sum of all the angles of the $\triangle$ having O as a vertex. | 3. Why? |
| 4. In the trihedral $\angle$ formed at A, B , etc., $\angle PAE + \angle PAB > \angle BAE$, or $\angle OAE + \angle OAB$; $\angle PBA + \angle PBC > \angle ABC$, or $\angle OBA + \angle OBC$, etc. | 4. § 75. |
| 5. $\therefore$ the sum of the angles at the bases of the $\triangle$ having P as a vertex $>$ the sum of the $\angle$ at the bases of the $\triangle$ having O as a vertex. | 5. § 317, XI. |
| 6. $\therefore$ the sum of the $\angle$ at the vertex $P <$ the sum of the angles at the vertex O . | 6. § 317, X. |

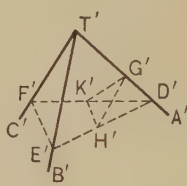
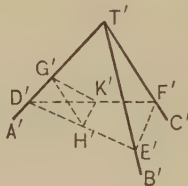
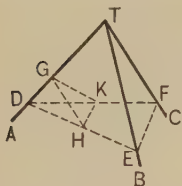
- ## Exercises

-

9. If a plane and a line not in it are both perpendicular to the same plane, they are parallel.

Proposition XX. Theorem

77. If two trihedral angles have the three face angles of one respectively equal to the three face angles of the other, the corresponding dihedral angles are equal.



Given: The trihedral $\angle T-ABC$ and $T'-A'B'C'$, having $\angle ATB = \angle A'T'B'$, $\angle BTC = \angle B'T'C'$, and $\angle ATC = \angle A'T'C'$.

To prove: The dihedral $\angle$ whose edges are TA , TB , and TC are equal respectively to the dihedral $\angle$ whose edges are $T'A'$, $T'B'$, and $T'C'$.

Plan of Attack: class — equal dihedral $\angle$.

method to be used — dihedral $\angle$ are equal if their plane angles are equal.

Proof:

1. On the edges of the trihedral $\angle$, lay off $TD = TE = TF = T'D' = T'E' = T'F'$. Draw DE , EF , DF , $D'E'$, $E'F'$, and $D'F'$.

2. $\triangle TDE = \triangle T'D'E'$, $\triangle TEF = \triangle T'E'F'$, and $\triangle TDF = \triangle T'D'F'$.

3. $\therefore DE = ?$, $EF = ?$, and $DF = ?$

4. $\therefore \triangle DEF = \triangle D'E'F'$.

5. At some point G in edge TD , draw GH in plane TDE and GK in plane TDF , each $\perp TA$, forming the plane $\angle HGK$ of the dihedral $\angle TA$.

1. §§ 309; 307.

2. Give proof.

3. § 322.

4. Why?

5. §§ 314; 309; 307.

On $T'D'$ take $D'G' = DG$ and draw the plane $\angle H'G'K'$ of the dihedral $\angle T'A'$. Draw HK and $H'K'$.

6. Right $\triangle DGH =$ right $\triangle D'G'H'$
and rt. $\triangle DGK =$ rt. $\triangle D'G'K'$.

7. $\triangle DHK = \triangle D'H'K'$.

8. $\triangle HGK = \triangle H'G'K'$.

9. $\therefore \angle HGK = \angle H'G'K'$.

10. $\therefore$ dihedral $\angle TA =$ dihedral $\angle T'A'$, and in a similar manner, dihedral $\angle TB$ and TC are proved equal respectively to dihedral $\angle T'B'$ and $T'C'$.

6. Give proof.

7. Give proof.

8. Give proof.

9. Why?

10. § 52.

Note: To fix this proof in mind the following order is easily remembered: five sets of congruent $\triangle$; large face $\triangle$, large base $\triangle$, small face $\triangle$, small base $\triangle$, then the $\triangle$ having the required plane $\triangle$.

Questions: What kind of triangle is $\triangle TDE$? Is $\angle TDE$ acute, right, or obtuse? Why? Can GH be parallel to DE ? Why? Why can we claim that GH meets DE ? Would $\triangle TDE = \triangle T'D'E'$, if TE and TD were unequal, and $TE = T'E'$ and $TD = T'D'$? Why then do we draw $TD = TE = TF =$ etc.?

78. Congruent, or equal, polyhedral angles are polyhedral angles which can be made to coincide; hence, they must have their corresponding parts equal and *arranged in the same order*.

79. Symmetrical polyhedral angles are polyhedral angles having their corresponding parts equal but *arranged in reverse order*. Which two figures of § 77 represent equal trihedral angles? Which two figures represent symmetric trihedral angles?

80. Corollary. — Two trihedral angles are congruent or symmetric, if the three face angles of one are equal respectively to the three face angles of the other.

Exercises

1. Two vertical trihedral angles are symmetrical.
2. An isosceles trihedral angle has two of its dihedral angles equal.
3. Find the projection of a rectangle 6×8 on the plane M , if a side 6 of the rectangle lies in the plane M and (a) the plane of the rectangle makes an angle of 30° with plane M ; (b) the plane of the rectangle makes an angle of 45° with plane M .
4. Given two parallel planes and a point between them. Through this point three lines not in the same plane are drawn intersecting each of the parallel planes. Prove that the intersections in each are the vertices of similar triangles.

True or False

Indicate which of the following are true statements (*i.e.* true in all cases) and which are false.

1. If a line is parallel to a plane, it is parallel to any line in the plane.
2. A line is perpendicular to a plane, if it is perpendicular to a line in the plane.
3. A plane is perpendicular to another plane, if it is perpendicular to a line in the other plane.
4. A line is parallel to a plane, if it is parallel to a line in the plane.
5. Two lines perpendicular to the same line are parallel.
6. Two lines parallel to the same line are parallel.
7. Two planes perpendicular to the same plane are parallel.
8. Two planes perpendicular to the same line are parallel.

BOOK TWO

POLYHEDRONS

Prisms

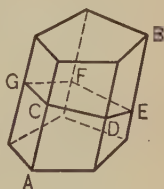
81. A **polyhedron** is a solid bounded by plane polygons.

The *faces* of the polyhedron are the bounding polygons; the *edges* are the intersections of the faces; the *vertices* are the intersections of the edges. A *diagonal* is a straight line joining two vertices not in the same face.

82. A **section** of a polyhedron is the figure formed by its intersection with a plane passing through it.

A *convex polyhedron* is a polyhedron in which every section is a convex polygon.

83. A **prism** is a polyhedron of which two faces are polygons in parallel planes and the other faces are parallelograms intersecting in parallel lines.

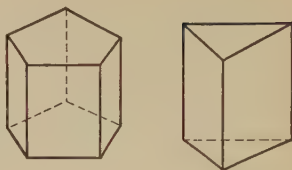


The *bases* of a prism are the polygons in the parallel planes; the *lateral faces* are the parallelograms included between the parallel planes; the *lateral edges* are the lines of intersection of the lateral faces; the *lateral area* is the sum of the areas of the parallelograms forming the lateral faces. The *altitude* of a prism is the perpendicular distance between the bases.

84. A **right section** of a prism is a section made by a plane perpendicular to the lateral edges; as right section *CDEFG*.

85. A **right prism** is a prism in which the lateral edges are perpendicular to the base.

An *oblique prism* is a prism in which the lateral edges are oblique to the base.



86. A **regular prism** is a right prism having regular polygons for its bases.

87. Prisms are **triangular**, **quadrangular**, or **hexagonal**, if their bases are triangles, quadrilaterals, or hexagons.

88. A **truncated prism** is the solid included between the base of a prism and a plane not parallel to the base cutting all the lateral edges.

Theorems Proved Informally

89. The lateral edges of a prism are equal.

90. The lateral edges of a right prism are altitudes.

91. The **volume of any solid** is the number of units of volume which it contains.

92. **Equivalent solids** are solids which have the same volume or size.

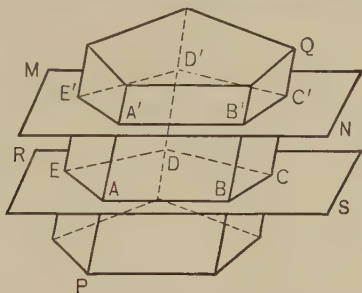
Exercises

1. Prove that any section of a prism made by a plane parallel to the lateral edges but not containing them is a parallelogram.

2. Prove that if a right section of a prism is an equilateral triangle, then every section which has one of its sides parallel to one of the sides of the given section is an isosceles triangle.

Proposition I. Theorem

93. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given: Prism PQ cut by the parallel planes RS and MN , forming the sections $ABCDE$ and $A'B'C'D'E'$.

Prove: $ABCDE = A'B'C'D'E'$.

Plan of Attack: class — congruent polygons.

method to be used — prove the corresponding sides, and corresponding angles, equal.

Proof:

- | | |
|--|---|
| 1. $AB \parallel ?$, $BC \parallel ?$, etc. | 1. Why?
2. § 42.
3. § 83.
4. Why?
5. § 397. |
| 2. $\therefore \angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc. | |
| 3. AA' , BB' , CC' , etc. are parallel. | |
| 4. $\therefore AB = A'B'$, $BC = B'C'$, etc. | |
| 5. $\therefore ABCDE = A'B'C'D'E'$. | |

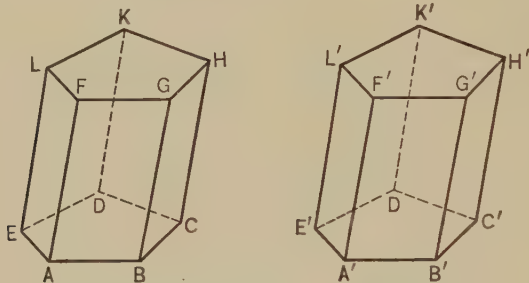
94. Corollary I. — The bases of a prism are congruent polygons.

95. Corollary II. — Every section of a prism parallel to the base is congruent to the base.

96. Corollary III. — All right sections of a prism are congruent polygons.

Proposition II. Theorem

97. *Two prisms are congruent if three faces which include a trihedral angle of one are respectively congruent to three faces which include a trihedral angle of the other, and are similarly placed.*



Given: Prisms AK and $A'K'$ with face $AD = \text{face } A'D'$, face $AL = \text{face } A'L'$, and face $AG = \text{face } A'G'$.

Prove: $AK = A'K'$.

Plan of Attack: class — congruent figures.

method to be used — show that they can be made to coincide throughout.

Proof:

- | | |
|--|-----------|
| 1. $\angle EAF = \angle E'A'F'$, $\angle FAB = \angle F'A'B'$, and $\angle EAB = \angle E'A'B'$. | 1. § 322. |
| 2. $\therefore$ trihedral $\angle A = \text{trihedral } \angle A'$. | 2. § 80. |
| 3. Place AK on $A'K'$ so that trihedral $\angle A$ coincides with trihedral $\angle A'$. | 3. § 389. |
| 4. Then faces AD , AL , and AG coincide with faces $A'D'$, $A'L'$, and $A'G'$, respectively. | 4. § 389. |
| 5. $\therefore$ points L , F , and G fall on points L' , F' , and G' ; and points C and D fall on points C' and D' . | 5. § 389. |
| 6. $\therefore$ plane LH coincides with plane $L'H'$. | 6. § 4. |

7. But polygon $LH =$ polygon $L'H'$.	7. §§ 94; 317, VII.
8. $\therefore LH$ coincides with $L'H'$ and points H and K fall on points H' and K' .	8. Why?
9. $\therefore CH$ and DK coincide with $C'H'$ and $D'K'$.	9. Why?
10. $\therefore AK = A'K'$.	10. § 389.

98. Corollary I. — Two truncated prisms are congruent, if three faces which include a trihedral angle of one are respectively congruent to three faces which include a trihedral angle of the other, and are similarly placed.

99. Corollary II. — Two right prisms having congruent bases and equal altitudes are congruent.

Exercises

1. What is the least number of faces that a polyhedron can have? a prism?

2. If a lateral edge of a prism is an altitude, what kind of prism is it?

3. Are the sides of a right section of a prism perpendicular to the lateral edges? Why?

4. Each lateral edge of a prism is 5'' and a right section is an equilateral triangle with side 3''. What is the lateral area of the prism?

5. A prism has a lateral edge 6'' long. Its right section is a regular hexagon with side 2'' long. Find the lateral area of the prism.

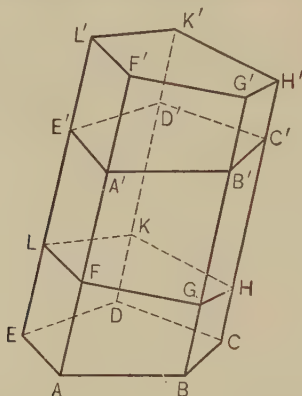
6. A prism has a square base. Is its right section necessarily a square?

7. If two prisms have congruent bases and equal altitudes, are the lateral edges necessarily equal?

8. If two prisms have congruent right sections and equal lateral edges, are the prisms congruent?

Proposition III. Theorem

100. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given: Oblique prism AD' with right section FK , and the right prism FK' with base FK and altitude or lateral edge equal to lateral edge of AD' .

Prove: $AD' = FK'$.

Plan of Attack: class — equivalent figures.

method to be used — prove truncated prism

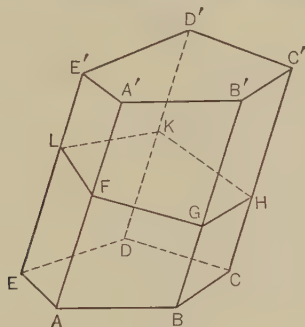
$AK =$ truncated prism $A'K'$; use Ax. 1.

Proof:

- | | |
|---|-----------------|
| 1. Edge $AA' =$ edge FF' . And $A'F = A'F'$. | 1. Why? |
| 2. $\therefore AF = A'F'$. Similarly $EL = ?$ | 2. § 317, II. |
| 3. Also $AE = A'E'$ and $LF = L'F'$. | 3. §§ 83; 333a. |
| 4. $\angle EAF = \angle E'A'F'$, $\angle AFL = ?$, etc. | 4. Why? |
| 5. $\therefore AL = A'L'$. Similarly $AG = A'G'$. | 5. § 397. |
| 6. Face $AD =$ face $A'D'$. | 6. § 94. |
| 7. $\therefore AK = A'K'$. | 7. § 98. |
| 8. $\therefore AK + FD' = A'K + FD'$. | 8. Why? |
| 9. $\therefore AD' = FK'$. | 9. Why? |

Proposition IV. Theorem

101. *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*



Given: Prism AD' with right section FK ; S its lateral area, e a lateral edge, and p the perimeter of FK .

To prove: $S = ep$.

Plan of Attack: class — areas of surfaces.

method to be used — show that the lateral area = the sum of the areas of the parallelograms forming the lateral faces.

Proof:

- | | |
|---|----------------------|
| 1. $AA' = BB' = CC' = \text{etc.} = e$. | 1. § 89. |
| 2. $FG \perp BB'$. | 2. §§ 84; 9c. |
| 3. $\therefore \text{area } \square AB' = BB' \times FG$
$= e \times FG$. | 3. § 366. |
| 4. Similarly $\text{area } \square BC' = e \times GH$, etc. | 4. Reasons 1-3. |
| 5. $\therefore S = e(FG + GH + \text{etc.})$. | 5. § 317, I and via. |
| 6. But $p = FG + GH + \text{etc.}$ | 6. Why? |
| 7. $\therefore S = ep$. | 7. Why? |

102. Corollary. — The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.

Exercises

1. The base of a right prism 50 centimeters high is a rhombus whose longer diagonal is 30 centimeters and whose shorter diagonal is 16 centimeters. Find the total surface of the prism.

2. Find the lateral area and the total area of a right prism whose altitude is 17 inches and whose base is an equilateral triangle with a side 5 inches.

3. Find the lateral area and the total area of a right prism whose altitude is 17 inches and whose base is a regular hexagon with a side 5 inches.

4. The right section of a prism is a rectangle $6'' \times 4''$. The lateral edge of the prism is $8''$. Find the lateral area.

5. The lateral edge of a prism is $10''$; a right section of the prism is an equilateral triangle with side $4''$. Find the lateral area.

6. A right section of a prism is a pentagon with a perimeter $42''$. The lateral edge is $14''$. Find the lateral area.

7. A truncated right prism has a square base with edge $4''$. Two adjacent lateral edges are each $9''$; the other two lateral edges are $6''$. Find the lateral area.

Parallelepipeds

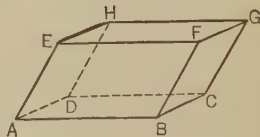
103. A **parallelepiped** is a prism whose bases are parallelograms.

104. A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to the bases.

An **oblique parallelepiped** is any parallelepiped which is not a right parallelepiped.

105. A **rectangular solid** is a right parallelepiped whose bases are rectangles.

106. A **cube** is a rectangular solid with all its edges equal.



Theorems Proved Informally

107. All the faces of a parallelepiped are parallelograms.
108. The lateral faces of a right parallelepiped are rectangles.
109. All the faces of a rectangular solid are rectangles.
110. All the faces of a cube are squares.
111. The opposite faces of a parallelepiped are congruent.
- In the figure (§ 103), $AE = BF$, $EH = FG$, $HD = CG$, and $AD = BC$. Why? $AE \parallel BF$, $EH \parallel FG$, $HD \parallel CG$, and $AD \parallel BC$. Why? $\therefore \angle EAD = \angle FBC$, $\angle ADH = \angle BCG$, etc. Why? $\therefore ADHE = BCGF$ (§ 397).
112. The opposite faces of a parallelepiped are parallel.

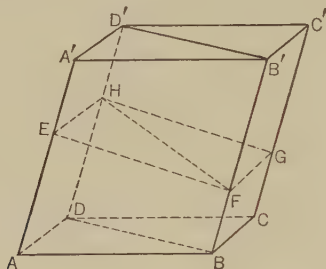
$AE \parallel BF$ and $AD \parallel BC$. Why? $\therefore$ plane $AH \parallel$ plane BG (§ 41).

Exercises

1. Prove that the square of a diagonal of any rectangular solid is equal to the sum of the squares of its three dimensions.
2. The dimensions of a rectangular solid are to be consecutive integers and the total surface is to be 94 square units. Find the dimensions.
3. Prove that the diagonals of a parallelepiped bisect one another.
4. Prove that any straight line drawn through the point of intersection of the diagonals of a parallelepiped and terminating in the opposite faces is bisected at that point.
5. Prove that if the diagonals of a quadrangular prism pass through a common point, the prism is a parallelepiped.
6. An open cistern is 4' 6'' long, 2' 8'' wide, and contains 42 cu. ft. How many square feet of tin will be required to line it?

Proposition V. Theorem

113. *The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.*



Given: Parallelepiped AC' with plane BD' passing through the edges DD' and BB' , forming the triangular prisms $ABD-A'$ and $BDC-C'$.

To prove: $ABD-A' = BDC-C'$.

Plan of Attack: class — equivalent solids.

method to be used — show that each is equal to a right prism by § 100, and that the right prisms are equal, etc.

Proof:

- | | |
|--|----------------|
| 1. Construct the right section EG cutting plane BD' in line HF . | 1. §§ 22; 10. |
| 2. $AD' \parallel ?$ and $AB' \parallel ?$ | 2. § 112. |
| 3. $\therefore EH \parallel ?$ and $EF \parallel ?$ | 3. § 12. |
| 4. $\therefore EFGH$ is a $\square$. | 4. § 319a. |
| 5. $\therefore \triangle EFH = \triangle FGH$. | 5. § 333d. |
| 6. $ABD-A' =$ a right prism with base EFH and altitude BB' , and $BDC-C' =$ a right prism with base FGH and altitude BB' . | 6. § 100. |
| 7. But the right prisms are equal. | 7. § 99. |
| 8. $\therefore ABD-A' = BDC-C'$. | 8. § 317, VII. |

Question: Are the two equivalent triangular prisms congruent?

Volumes

114. Measurement of Volumes. — In plane geometry we learned that a line is measured by finding how many times it contains a unit line segment; an angle, by finding how many times it contains a unit angle; the area of any figure, by finding how many times the inclosed surface contains a unit surface. Likewise, we measure the volume of any solid by finding how many times a unit of volume is contained within the solid.

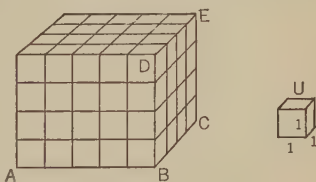
115. The volume of any solid is the number of times it contains a unit of volume. In other words, it is the number which expresses its numerical measure in terms of the unit of volume.

116. The unit of volume is a cube each of whose edges is a unit length; as, a cube each edge of which is 1 foot in length, or 1 inch in length.

117. Measurement Illustrated. — In the accompanying figure the rectangular solid AE contains the unit cube U , 80 times; hence the volume of the solid is 80.

It is evident also that the length AB of the rectangular solid contains one edge of the unit cube 5 times, the width BC contains one edge of the unit 4 times, and the height BD contains one edge of the unit 4 times; hence, the solid is divided into 4 layers, or groups, each containing the unit volume 5×4 or 20 times. Hence, the volume of the rectangular solid 80 equals $5 \times 4 \times 4$.

If the rectangular solid is of such dimensions that no cube can be found having an edge which is exactly contained in



each of the edges AB , BC , and BD , we can find the volume of the solid approximately by taking the edge of the unit cube sufficiently small. The approximate volume of the solid will then equal the product of the approximate measurements of the three edges representing its length, breadth, and height.

These inferences illustrate the theorem :

118. *The volume of a rectangular solid is equal to the product of its three dimensions.* Stated as a formula, $V = abc$.

This statement may be taken as the definition of the volume of a rectangular solid, and the four theorems, §§ 120, 122, 124, and 125, may then be omitted or proved algebraically.

Since the product of the length and breadth is the area of the base (§ 366), we may state the above theorem as follows :

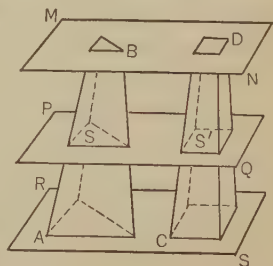
The volume of a rectangular solid is equal to the product of the area of its base and its altitude; or, $V = bh$.

It should be remembered that *the volume of any solid is expressed as a number*, which represents the *numerical measure* of the solid. The product of the base and altitude means the product of their numerical measures.

119. Cavalieri's Theorem.—The following theorem, proved formally in later mathematics, is here assumed. It may be used as a basis for comparing the volumes of various solids.

If two solids contained between the same parallel planes are such that their sections by a plane parallel to those planes are equal in area, the two solids have the same volume.

In the figure the solids AB and CD are contained between the parallel planes MN and RS . If the



sections s and s' formed by *any* plane PQ parallel to MN and RS are equal, then the volume of AB = the volume of CD .

We may infer the truth of this statement by imagining two material solids, with their bases in the same parallel planes, to be cut by a slicing machine (such as is used in markets). Since these slices, parallel to the bases, may be made as thin as we please, and since each slice of one has the same area as the corresponding slice of the other, the volumes of any two corresponding slices may be considered as equal. Since the two solids are contained between two parallel planes, they will consist of the same number of slices, which are equal in pairs. Hence, the volumes of the solids will be equal.

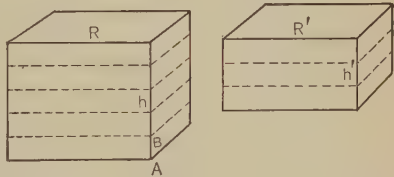
Note: If the teacher prefers, § 130 may be proved by using this theorem as is done in § 127. In this case § 130 would include the statements of §§ 127 and 129, which could then be omitted.

120. Theorem. — *The volumes of two rectangular solids having equal bases are to each other as their altitudes.*

Let R and R' be two rectangular solids with equal bases, and altitudes h and h' .

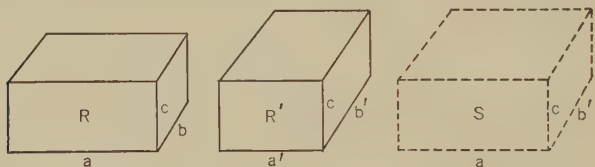
Suppose a common measure of h and h' is contained 5 times in h and 3 times in h' . Then $h : h' = 5 : 3$. Through the points of division pass planes parallel to the bases. Then R and R' will be divided into equal right prisms, 5 of which will be contained in R and 3 in R' . Then $R : R' = 5 : 3$. $\therefore R : R' = h : h'$.

Note: If the statement of § 118 is taken as the definition of the volume of a rectangular solid, prove §§ 120, 122, and 124 algebraically. The theorem in § 125 may then be omitted.



121. Corollary. — If two rectangular solids have two dimensions in common, they are to each other as their third dimensions.

122. Theorem. — *The volumes of two rectangular solids having equal altitudes are to each other as their bases.*

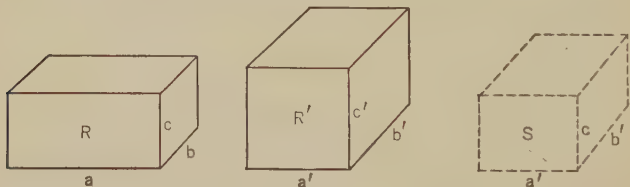


Let R and R' be two rectangular solids with common altitude c , and with the dimensions of their bases a, b and a', b' .

Construct a third rectangular solid S , with dimensions of its base a and b' and with altitude c . Then $R:S = b:b'$ and $R':S = a':a$ (§ 121). $\therefore R:R' = ab:a'b'$.

123. Corollary. — Two rectangular solids which have one dimension in common are to each other as the products of their other two dimensions.

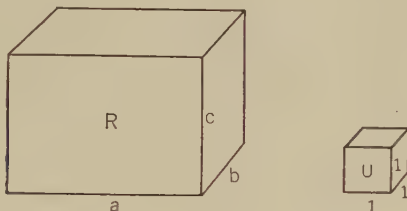
124. Theorem. — *The volumes of two rectangular solids are to each other as the products of their three dimensions.*



Let R' and R' have the dimensions a, b, c and a', b', c' . Construct a third rectangular solid with two dimensions of one and the third dimension of the other. Then $R:S = ab:a'b'$ (§ 123) and $R':S = c':c$ (§ 121). $\therefore R:R' = abc:a'b'c'$.

Proposition VI. Theorem

125. *The volume of a rectangular solid is equal to the product of its three dimensions.*



Given: Rectangular solid R with its three dimensions a , b , and c .

To prove: The volume of $R = abc$.

Plan of Attack: class — comparison of volumes.

method to be used — find the ratio of R to a unit of volume by using § 124.

Proof:

- | | |
|--|----------------|
| 1. Let U represent the unit of volume, with each of its dimensions some linear unit. | 1. § 116. |
| 2. Then $\frac{R}{U} = \frac{a \times b \times c}{1 \times 1 \times 1} = abc$. | 2. § 124. |
| 3. But the volume of $R = \frac{R}{U}$. | 3. § 115. |
| 4. $\therefore$ the volume of $R = abc$. | 4. § 317, VII. |

Note: See § 118. It should be remembered that the product of lines means the product of their numerical measures. The three dimensions of a rectangular solid are understood to be the measurements of two adjacent sides of the rectangular base and the altitude or lateral edge.

126. Corollary. — The volume of a rectangular solid is equal to the product of its base and its altitude; or, $V = bh$.

127. Corollary. — The volume of any parallelepiped is equal to the product of its base and its altitude. (See Note, § 119.)

Extend the planes of the bases of the parallelepiped. Construct a rectangular solid with its bases in these planes, making its base equal in area to the base of the given solid. Then the two solids will have equal bases and the same altitude (§ 35). Sections of the two solids, formed by *any* plane parallel to the planes of the bases, will be equal in area (§ 95 and Ax. VII). $\therefore$ the volume of the parallelepiped = the volume of the rectangular solid (§ 119); or, $V = bh$.

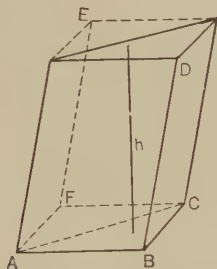
128. Corollary. — The volume of a cube is the cube of its edge.

Exercises

1. The total surface of a cube is 450 square inches. Find its volume.
2. The entire surface of a cube is 1014 sq. ft. Find the volume of the cube and the length of one diagonal.
3. The diagonal of a cube is 10 inches. Find the volume of the cube.
4. The edge of a cube is e , its total surface is s , and its volume v . (a) Find s and v in terms of e . (b) Find s in terms of v .
5. The diagonal of a cube is $7\sqrt{3}$. Find its volume and its total surface.
6. The dimensions of a rectangular solid are 8'', 10'', and 12''. Find the total surface and the volume of the solid.
7. A diagonal of a rectangular solid is $10\sqrt{2}$ ''. The edges of the base are 8'' and 6''. Find the volume of the solid.
8. Each lateral edge of a parallelepiped is 8''. A right section is a square with side 4''. What is the volume of the parallelepiped?
9. Is the lateral edge of a parallelepiped necessarily an altitude?

Proposition VII. Theorem

129. *The volume of a triangular prism is equal to the product of its base and its altitude.*



Given: Triangular prism $ABC-D$ with base area b , altitude h , and volume V .

To prove: $V = bh$.

Plan of Attack: class — volume of a solid.

method to be used — show that the triangular prism is equal to one half the volume of a parallelepiped having the same altitude and having a base equal to twice the base of the given prism.

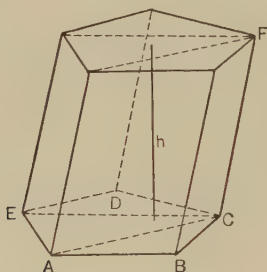
Proof:

- | | |
|---|-------------------|
| 1. Construct the parallelepiped BE whose edges are parallel to AB , BC , and BD . | 1. §§ 4; 12; 313. |
| 2. Then prism $ABC-D =$ prism $AFC-E$ and $\triangle ABC = \triangle AFC$. | 2. §§ 113; 333d. |
| 3. $\therefore 2 ABC-D = BE$ and $2b = ABCF$. | 3. Why? |
| 4. But volume of $BE = ABCF \times h = 2bh$. | 4. § 127. |
| 5. $\therefore 2 ABC-D = 2bh$. | 5. Why? |
| 6. $\therefore$ volume of $ABC-D$, or V , $= bh$. | 6. § 317, iv. |

Note: See § 119, Note.

Proposition VIII. Theorem.

130. *The volume of any prism is equal to the product of its base and its altitude.*



Given: Prism AF with base area b , altitude h , and volume V .

To prove: $V = bh$.

Plan of Attack: class — volume of solid.

method to be used — show that the volume of the given prism equals the sum of the volumes of the triangular prisms whose bases are triangles formed by drawing diagonals of the given base.

Proof:

- | | |
|--|-----------|
| 1. From vertex C of the base AD draw diagonals AC , EC , etc. | 1. § 307. |
| 2. Through edge CF and the diagonals AC , EC , etc., pass planes. | 2. § 4. |
| 3. Then AF is divided into triangular prisms which have the altitude h . | 3. Why? |
| 4. Then volume $ABC-F = ABC \times h$,
volume $EAC-F = CAE \times h$, etc. | 4. § 129. |
| 5. $\therefore$ the sum of volumes of the triangular prisms $= (ABC + EAC + \text{etc.})h$. | 5. Why? |
| 6. $\therefore$ volume prism $AF = bh$
or $V = bh$. | 6. Why? |

Note: See § 119, Note.

Exercises

1. The volume of a regular hexagonal prism is $81\sqrt{3}$; the altitude of the prism is equal to the longest diagonal of the base. Find the total area of the prism.

2. Find the number of cubic feet of earth in a railway embankment 2500 feet long, 10 feet high, 12 feet wide at the top, and 42 feet wide at the bottom.

3. A right prism has a rhombus for its base. If the diagonals of this base are 12 feet and 16 feet, and the height of the prism is 20 feet, find the total surface and the volume of the prism.

4. A watering trough in the form of a regular triangular prism has one of its lateral edges on the ground and the opposite face horizontal. It is filled with water to a depth of two thirds of the altitude of the trough in this position. What part of the volume of the trough is occupied by the water?

5. Given a truncated prism with square base $ABCD$ and lateral edges AE , BF , CG , and DH , perpendicular to the base. If $AB = 10$, $AE = BF = 6$, $CG = DH = 10$, (a) prove $EFGH$ is a rectangle; (b) find DG and DF ; (c) find the area of each face; (d) find the volume of the truncated prism.

6. The base of a prism is a regular hexagon with each side equal to $3''$. The lateral edges are $8''$ in length and make an angle of 60° with the base. What is the volume of the prism?

7. A right section of a prism is an equilateral triangle with side $4''$. Each lateral edge of the prism is $6''$. What is the volume of the prism?

8. Two regular prisms, one with a triangular base and the other with an hexagonal base, have equal altitudes and equal bases. If the area of each base is $36\sqrt{3}$ and the lateral edge 10, what is the volume of each solid? What is the lateral area of each solid?

9. If two prisms have congruent bases and equal altitudes, are they necessarily equal in volume? Are they necessarily congruent? If not congruent, are their lateral areas equal?

Pyramids

131. A **pyramid** is a polyhedron of which one face is a polygon and the other faces are triangles having the sides of the polygon as bases and having a common vertex.

The *base* of the pyramid is the polygon; the *lateral faces* are the triangles; the *lateral edges* are the intersections of the lateral faces; the *vertex* is the common vertex of the triangles; the *altitude* is the perpendicular from the vertex to the plane of the base.



132. A **triangular pyramid** is a pyramid having a triangle for its base; a *quadrangular pyramid* has a quadrilateral for its base; etc.

A *tetrahedron* is a polyhedron having four faces. Hence, a triangular pyramid is a tetrahedron.

133. A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude meets the center of the base.

A *regular tetrahedron* has all its faces equilateral triangles.

By § 29 the lateral edges of a regular pyramid are equal. Hence, the lateral faces are congruent isosceles triangles, which have equal altitudes.

134. The **slant height** of a regular pyramid is the altitude of any one of its lateral faces.

135. A **frustum of a pyramid** is the portion of the pyramid included between its base and a plane parallel to the base.

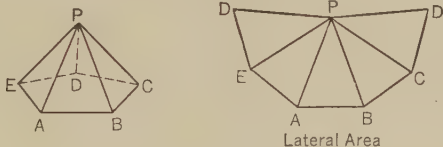
The lateral faces of a frustum are trapezoids (§ 12). The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.



136. The **altitude** of a frustum of a pyramid is the perpendicular between the planes of the bases.

137. The **slant height** of a frustum of a regular pyramid is the altitude of any one of its lateral faces.

138. The **lateral surface**, or **lateral area**, of a pyramid is the sum of the areas of its lateral faces.



Exercises

1. Find the altitude of a triangular pyramid every edge of which is 10 inches.

2. Prove that any section of a tetrahedron made by a plane parallel to two opposite edges is a parallelogram.

3. In the pyramid $A-BCD$, prove that the lines joining in order the midpoints of BC , AC , AD , and BD form a parallelogram.

4. A regular pyramid with a square base has each of its eight edges 4 inches. Find the height of the pyramid.

5. Prove that the three lines joining the midpoints of the opposite edges of a tetrahedron bisect one another, hence meet in a point.

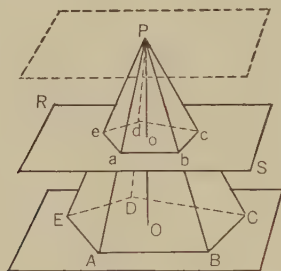
6. Find the altitude of a regular hexagonal pyramid, if each edge of the base is 6'' and each lateral edge 10''.

7. Find the lateral area of a triangular pyramid every edge of which is 8 inches.

8. A triangular pyramid has for its base an isosceles right triangle ABC ; $\angle ACB$ being a right angle. The lateral edge CD is perpendicular to the base. If $AC = CB = CD = 4''$, find the lateral area.

Proposition IX. Theorem

139. *If a pyramid is cut by a plane parallel to the base, the lateral edges and the altitude are divided proportionally, and the section is a polygon similar to the base.*



Given: Pyramid $P-ABCDE$ and plane RS cutting the lateral edges in a, b, c, d , and e ; altitude PO cuts RS in o .

To prove: I. $\frac{Pa}{PA} = \frac{Pb}{PB} = \dots = \frac{Po}{PO}$ and

II. Section $abcde \sim ABCDE$.

Plan of Attack: class — proportional lines; similar polygons.
 methods to be used — parallel planes cut off proportional segments on two lines; show that the polygons are mutually equiangular and that their corresponding sides are proportional.

Proof:

- | | |
|--|-----------|
| I. 1. Through vertex P pass plane parallel to base. | 1. § 39. |
| 2. Then $\frac{Pa}{PA} = \frac{Pb}{PB} = \dots = \frac{Po}{PO}$. | 2. § 44. |
| II. 3. $ab \parallel AB, bc \parallel BC$, etc. | 3. Why? |
| 4. $\therefore \angle abc = \angle ABC, \angle bcd = \angle BCD$, etc. | 4. § 42. |
| 5. $\triangle Pab \sim \triangle PAB, \triangle Pbc \sim \triangle PBC$, etc. | 5. § 362. |

- | | |
|---|-----------|
| 6. $\therefore \frac{ab}{AB} = \frac{Pb}{PB} = \frac{bc}{BC} = \frac{Pc}{PC} = \frac{cd}{CD} = \text{etc.}$ | 6. § 398. |
| 7. $\therefore \frac{ab}{AB} = \frac{bc}{BC} = \frac{cd}{CD} = \text{etc.}$ | 7. Why? |
| 8. $\therefore \quad \quad \quad abcde \sim ABCDE.$ | 8. § 398. |

140. Corollary I. — Any section of a pyramid parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude.

In the figure for § 139, $abcde \sim ABCDE$. $\therefore \frac{abcde}{ABCDE} = \frac{\overline{ab}^2}{\overline{AB}^2}$
 (§ 369). But $\frac{ab}{AB} = \frac{Po}{PO}$ (statements 6 and 2; Ax. VIII).
 $\therefore \frac{\overline{ab}^2}{\overline{AB}^2} = \frac{\overline{Po}^2}{\overline{PO}^2}$ (Ax. V). $\therefore \frac{abcde}{ABCDE} = \frac{Po^2}{PO^2}$ (Ax. VIII).

141. Corollary II. — If two pyramids have equivalent bases and equal altitudes, sections of the pyramids made by planes parallel to the bases at equal distances from the vertices are equivalent.

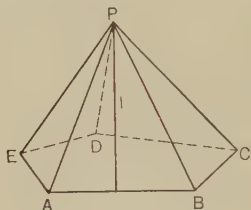
Let $P-ABCDE$ and $P'-A'B'C'$ be two pyramids with base $ABCDE = \text{base } A'B'C'$ and altitude $PO = \text{altitude } P'O'$. Sections $abcde$ and $a'b'c'$ cut by PO and $P'O'$ in points o and o' . $Po = P'o'$. Then $\frac{abcde}{ABCDE} = \frac{Po^2}{PO^2}$ and $\frac{a'b'c'}{A'B'C'} = \frac{P'o'^2}{P'O'^2}$ (§ 140). But $Po = P'o'$ and $PO = P'O'$. $\therefore \frac{abcde}{ABCDE} = \frac{a'b'c'}{A'B'C'}$. Why?
 $\therefore abcde = a'b'c'$ (§ 359).

142. Corollary III. — Two pyramids having equivalent bases and equal altitudes are equivalent.

Place the pyramids so that their bases are in the same plane and their vertices in a plane parallel to the base plane. Prove by using § 141 and Cavalieri's theorem.

Proposition X. Theorem

143. *The lateral area of a regular pyramid is equal to half the product of its slant height and the perimeter of the base.*



Given: Regular pyramid $P-ABCDE$, with S the lateral area, l its slant height, and p the perimeter of the base.

To prove: $S = \frac{1}{2} lp$.

Plan of Attack: class — areas.

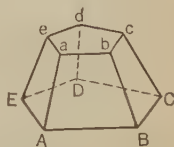
method to be used — the lateral area equals the sum of the areas of the triangles forming the lateral surface.

Proof:

- | | |
|---|-----------|
| 1. l is the altitude of each of the $\triangle$ forming the lateral surface. | 1. § 134. |
| 2. $\therefore$ area $\triangle PAB = \frac{1}{2} l \cdot AB$,
area $\triangle PBC = \frac{1}{2} l \cdot BC$, etc. | 2. § 367. |
| 3. $\therefore$ sum of areas of $\triangle = \frac{1}{2} l(AB + BC + \text{etc.})$. | 3. Why? |
| 4. $\therefore$ lateral area $S = \frac{1}{2} lp$. | 4. Why? |

144. Corollary. — The lateral area of a frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum; or $S = \frac{1}{2} l(p + p')$.

Show that the area of trapezoid $ABba = \frac{1}{2} l(AB + ab)$ (§ 368), area $BCcb = \frac{1}{2} l(BC + bc)$, etc.



Exercises

1. The area of the lower base of a frustum of a pyramid is 5 square inches, the area of the upper base is 4 square inches, and the altitude of the frustum is 2 inches. Find the altitude of the completed pyramid.

2. The altitude of the frustum of a regular square pyramid is 12 in. and the areas of the bases are 64 sq. in. and 16 sq. in. respectively. Find the slant height of the frustum.

3. A frustum of a pyramid is cut from a pyramid the perimeter of whose base is 60 inches and whose altitude is 15 inches. What is the altitude of the frustum if the perimeter of the upper base is 40 inches?

4. The base of a regular pyramid 8 inches high is a hexagon whose side is 6 inches. Find the entire surface of the pyramid.

5. Given the base edge a and the total surface t of a regular pyramid with a square base. Find the height h .

6. If m is the perimeter of a section bisecting all the lateral edges of a pyramid, show that $S = lm$.

7. Find the total surface of a regular pyramid having each side of the base 10 and lateral edge 13, if the base is (a) a triangle; (b) a square; (c) a hexagon.

8. In the pyramid $V-ABCD$, the base $ABCD$ is a square with edge 6, the lateral edge VA is perpendicular to the base and is 8 in length. Find the lateral area.

9. A regular square pyramid has each edge of the base 12 and altitude 8. Find the lateral area of the pyramid.

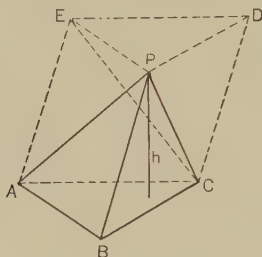
10. Find the total surface of a regular tetrahedron if an edge is 12".

11. A pyramid with altitude 9" is cut by a plane parallel to the base and 3" from the base. Find the ratio of the lateral areas of the frustum and the pyramid.

12. With the aid of the tables on pages 177-179 find the total surface of a regular pentagonal (and octagonal) pyramid, if a side of the base is 8" and the lateral edge 16".

Proposition XI. Theorem

145. *The volume of a triangular pyramid is equal to one third the product of its base and its altitude.*



Given: Triangular pyramid $P-ABC$ with base area b , altitude h , and volume V .

To prove: $V = \frac{1}{3}bh$.

Plan of Attack: class — volume of solids.

method to be used — show that $P-ABC$ is one of three equivalent pyramids whose sum equals the volume of a triangular prism whose base is b and altitude h .

Proof:

- | | |
|--|------------------|
| 1. Through A and C draw AE and CD equal and parallel to PB . Through E , P , and D pass a plane. | 1. §§ 4; 313. |
| 2. BE , BD , and AD are $\square$ and $EP \parallel AB$ and $PD \parallel BC$. | 2. §§ 339c; 390. |
| 3. $\therefore$ plane $EPD \parallel$ plane ABC . | 3. § 41. |
| 4. $\therefore D-ABC$ is a triangular prism with base b and altitude h . | 4. §§ 83; 87. |
| 5. Pass a plane through PE and PC , cutting face AD in line CE . | 5. §§ 4; 10. |
| 6. Then $\triangle AEC = \triangle EDC$ and pyramids $P-AEC$ and $P-EDC$ have the same altitude and equal bases. | 6. §§ 333d; 26. |

- | | |
|--|------------------------------|
| 7. $\therefore P-AEC = P-EDC$. | 7. § 142. |
| 8. $\triangle ABC = \triangle EPD$. | 8. Why? |
| 9. $\therefore$ pyramids $P-ABC$ and $C-EPD$ | 9. Why? |
| or $P-EDC$ have equal bases and their | |
| vertices P and C in parallel planes. | |
| 10. $\therefore P-ABC = C-EPD$ | 10. §§ 35; 142. |
| or $P-EDC$. | |
| 11. $\therefore P-ABC = P-EDC = P-AEC$. | 11. Why? |
| 12. $\therefore 3 P-ABC = \text{prism } D-ABC$. | 12. § 317, <i>via</i> ; VIII |
| 13. But volume $D-ABC = bh$. | 13. § 129. |
| 14. $\therefore$ volume $P-ABC = \frac{1}{3}bh$, or | 14. Why? |
| $V = \frac{1}{3}bh$. | |

Exercises

1. Find the volume of a regular tetrahedron each of whose edges is 6 inches.

2. Find the volume of a regular tetrahedron whose edge is $3\sqrt{2}$.

3. $ABCD$ is a square whose area is 36 square inches. It is folded about AC until the plane of the triangle ABC is perpendicular to that of triangle ADC . (a) Prove that the tetrahedron $BADC$ has two of its faces equilateral triangles. (b) Calculate the volume of this tetrahedron.

4. Derive the formula for the total surface (T), the altitude (H), and the volume (V) of a regular tetrahedron whose edge is a .

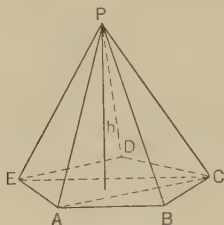
5. The total surface (T) of a regular tetrahedron is $100\sqrt{3}$ square units. Find the altitude (H) and the volume (V) of the solid.

6. The corner of a cube is cut off by a plane passed through the outer extremities of the three edges, meeting at the given corner. What part of the volume of the cube is thus removed?

7. Each base edge of a regular triangular pyramid is 6'' and the altitude of the pyramid is 4''. Find the lateral edge, the lateral area, and the volume.

Proposition XII. Theorem

146. *The volume of any pyramid is equal to one third the product of its base and its altitude.*



Given: Pyramid $P-ABCDE$ with base area b , altitude h , and volume V .

To prove: $V = \frac{1}{3}bh$.

Plan of Attack: class — volume of a solid.

method to be used — show that the volume of the pyramid equals the sum of the volumes of triangular pyramids having the same altitude and the sum of whose bases equals the base of the given pyramid.

Proof:

- | | |
|--|---------------|
| 1. Through P and the diagonals AC and CE drawn from any vertex of the base, pass planes dividing the pyramid into triangular pyramids. | 1. §§ 4; 307. |
| 2. Volume $P-ABC = \frac{1}{3}ABC \times h$,
volume $P-AEC = \frac{1}{3}AEC \times h$, etc. | 2. § 145. |
| 3. $\therefore$ sum of volumes of triangular pyramids
$= \frac{1}{3}(ABC + AEC + \text{etc.})h$. | 3. Why? |
| 4. $\therefore$ volume $P-ABCDE = \frac{1}{3}bh$, or $V = \frac{1}{3}bh$. | 4. Why? |

147. Corollary. — The volume of a frustum of any pyramid is expressed by the formula, $V = \frac{1}{3}h(b + b' + \sqrt{bb'})$, where b and b' are the areas of the bases and h the altitude of the frustum.

This can be proved by completing the pyramid and subtracting the volume of the small pyramid, whose base is b' and altitude x , from the volume of the whole pyramid, whose base is b and altitude $(x + h)$. $V = \frac{1}{3}(x + h)b - \frac{1}{3}xb' = \frac{1}{3}hb + \frac{1}{3}x(b - b')$.

$$\text{But } \frac{x^2}{(x + h)^2} = \frac{b'}{b} \text{ (§ 140). } \therefore x = \frac{h\sqrt{bb'} + hb'}{b - b'}.$$

$$\therefore V = \frac{1}{3}hb + \frac{1}{3}\left(\frac{h\sqrt{bb'} + hb'}{b - b'}\right)(b - b') \text{ (Ax. VIII), etc.}$$

Exercises

1. A regular triangular pyramid has each lateral edge equal to 5 and each base edge equal to 6. Find its volume and its total surface.

2. Find the volume of a regular triangular pyramid if each edge of the base is $10''$ and each lateral face makes an angle of 60° with the base.

3. $D-ABC$ is a tetrahedron with lateral edge DA perpendicular to the plane of the base ABC . ABC is an isosceles triangle. If $AB = AC = 5$, $BC = 6$, and $DA = 4$, (a) how large is the dihedral $\angle D-BC-A$? (b) What is the total surface of the tetrahedron? (c) What is the volume of the tetrahedron?

4. DA is a lateral edge and the isosceles $\triangle ABC$ is the base of a right prism. $AB = AC = 10$, $BC = 12$, and $DA = 11$. Point E is taken on DA , so that $AE = 8$. Through E and BC a plane is passed. (a) Find the number of degrees in the plane angle of the dihedral $\angle E-BC-A$. (b) Find the volume of the pyramid $E-ABC$. (c) Find the perpendicular distance from A to the plane EBC .

5. Given a rectangular solid with base $ABCD$ and lateral edge EA ; plane determined by point E and diagonal BD ; $EA = 12$, $AB = 20$, and $AD = 15$. (a) Show that the dihedral $\angle E-BD-A = 45^\circ$. (b) Find the volume of $E-ABD$. (c) Find the perpendicular distance from A to the plane EBD .

Exercises

1. The altitude of a frustum of a pyramid is 7.2 inches; the lower base is 10 inches square and the upper base 4 inches square. Find the volume of the frustum.

2. The base of a regular pyramid 8 inches high is a hexagon whose side is 6 inches. Find the volume of the pyramid.

3. The volume of a regular square pyramid is 18 cubic feet; its altitude is twice one side of the base. Find (a) the total surface of the pyramid, (b) the area of a section made by a plane parallel to the base and one foot from the base.

4. Find the volume of a pyramid whose base contains 30 sq. in., if one lateral edge is 5 in. and the angle formed by this edge and the plane of the base is 45° .

5. Prove that the volume of any regular pyramid is equal to one third its lateral area multiplied by the perpendicular distance from the center of its base to any lateral face.

6. A square pyramid has each edge of its base equal to m and each lateral edge equal to 10. Express the volume in terms of m .

7. If a square pyramid has each edge of the base equal to x and each lateral edge equal to m , show that the volume is $\frac{x^2}{6}\sqrt{2(m^2 - x^2)}$.

8. On the six faces of a cube as bases regular pyramids are constructed, all lying without the cube. The height of each pyramid is one half the edge e of the cube. Find (a) the surface of the entire solid, (b) the volume of the entire solid.

9. A house is 30 feet square and 25 feet high; its roof has the form of a square pyramid 10 feet high. Find the entire surface and the volume of the house.

10. The frustum of a regular pyramid has square bases 8'' and 4'' respectively on a side, and an altitude of 15''. Find the altitude of an equivalent pyramid whose base is a mid-section of the frustum.

11. A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 4 ft. and 3 ft. respectively, surmounted by a regular quadrangular pyramid 3 ft. in height, each side of whose base is 3 ft. What is the weight of the monument at 180 lb. to the cubic foot?

12. Find the lateral area and the volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17'' and 7'' respectively, and whose altitude is 12''.

13. Find the lateral area and the volume of the frustum of a regular hexagonal pyramid, the sides of whose bases are 20 feet and 10 feet respectively, and whose altitude is 16 feet.

14. Find the lateral area and the volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 feet and 6 feet respectively, and whose altitude is 5 feet.

15. Find the volume of a frustum of a pyramid with altitude 10 if the upper and lower bases are rhombuses with edges 6 and 8 respectively, and if one angle in each base is 60° .

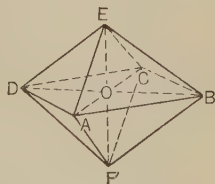
16. Find the volume of a regular pyramid having each side of the base 10'' and the lateral edge 20'', if the base is (a) a triangle; (b) a square; (c) a hexagon.

17. With the aid of the tables on pages 177-179 find the volume of a regular pentagonal (and a regular octagonal) pyramid, if a side of the base is 10'' and the lateral edge is 20''.

18. The base of a pyramid is a rectangle $ABCD$. The lateral edge VA is perpendicular to the base. $AB = 6$, $BC = 8$, and $VA = 10$. Find the total area and the volume.

19. Find the total surface and the volume of a frustum of the pyramid, exercise 18, cut off by a plane 6'' from the base.

20. In the accompanying figure the rhombus $ABCD$ and the rhombus $EAFC$ are in perpendicular planes intersecting in AC . $DB = 16$, $AC = 12$, and $EF = 15$. Find the volume of the octahedron.



21. Find the total surface and the volume of a frustum of a regular triangular pyramid, if the side of the lower base is 10, the side of the upper base 5, and the lateral edge 10.

22. If the side of the lower base is 7, the side of the upper base is 3, and the altitude is 4, find the total surface and the volume of the frustum of a regular pyramid with triangular base; with square base; with hexagonal base.

23. In the figure $ABCD A' B' C' D'$ is a cube of edge 12 and P is the midpoint of $A' B'$.

(a) Find PA , PB , PC , and PD .

(b) What angle is a plane angle of the dihedral formed by $ABCD$ with PBC ? with PAD ?

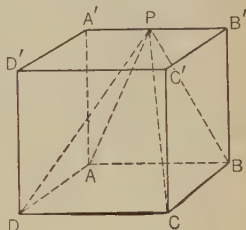
(c) Explain a method for constructing a plane angle of the dihedral angle formed by $ABCD$ with PCD . How many degrees are there in this angle?

(d) Find the total surface of the pyramid $P-ABCD$.

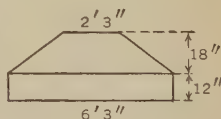
(e) Find the volume of the pyramid $P-ABCD$.

(f) Find the total surface of a frustum of $P-ABCD$ formed by a plane at a distance 6 from the base.

(g) Find the volume of the same frustum.



24. The concrete base of a column is square and has the dimensions shown in the accompanying figure. How many cubic yards of concrete are there in 10 such pedestals?



25. Given a cube with edge equal to 6 and base $ABCD$ and lateral edges AE , BF , CG , and DH . Through J , K , and L , the midpoints of EF , BF , and FG , respectively, a plane is passed and the pyramid $F-JKL$ is removed. In a similar manner pyramids are removed at each of the other vertices of the cube. Find the total surface and the volume of the remaining solid.

26. If the above cube is cut by the planes BEG , BED , BDG , and EDG , find the total surface of the resulting solid $BDEG$.

Similar Polyhedrons

148. Similar polyhedrons are polyhedrons that have the same number of faces, respectively similar and similarly placed, and have their corresponding polyhedral angles congruent.

149. Theorems proved informally. The following statements may be assumed, or proved informally:

(a) The corresponding edges of similar polyhedrons are proportional.

(b) Any two corresponding lines in two similar polyhedrons have the same ratio as any two corresponding edges.

(c) Two corresponding faces of similar polyhedrons are proportional to the squares of any two corresponding edges.

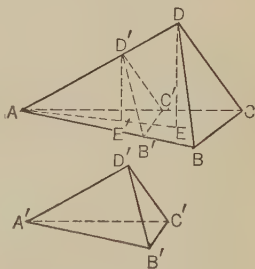
(d) The areas of two similar polyhedrons are proportional to the squares of any two corresponding edges.

150. Theorem. — The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.

Outline of proof: Trihedral $\angle A$ and A' are congruent (§ 148). Apply $D'-A'B'C'$ to $D-ABC$ so that $\angle A'$ coincides with $\angle A$.

Pass a plane through $AD \perp$ plane ABC (§ 63).

Draw DE and $D'E' \perp AE$ and $\therefore \perp$ plane ABC (§ 59).



$$\frac{V}{V'} = \frac{\triangle ABC \times DE}{\triangle A'B'C' \times D'E'} = \frac{\triangle ABC}{\triangle A'B'C'} \times \frac{DE}{D'E'}$$

(§ 146 and Ax. IV).

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB^2}{A'B'^2} \text{ (§ 149c) and } \frac{DE}{D'E'} = \frac{AB}{A'B'} \text{ (§ 149b).}$$

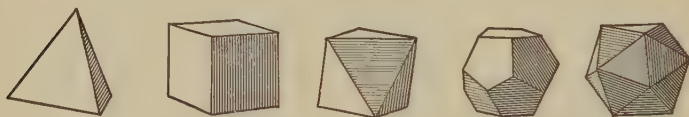
$$\therefore \frac{V}{V'} = \frac{AB^3}{A'B'^3}.$$

151. Theorem. — The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.

Assume that the polyhedron can be divided into tetrahedrons similar each to each and similarly placed and use §§ 150 and 149a, together with axioms.

Regular Polyhedrons

152. A **regular polyhedron** is a polyhedron whose faces are congruent regular polygons, and whose polyhedral angles are congruent.



153. Theorem. — There cannot be more than five regular polyhedrons.

A polyhedral angle must have at least three faces (§ 72), and the sum of its face angles must be less than 360° (§ 76).

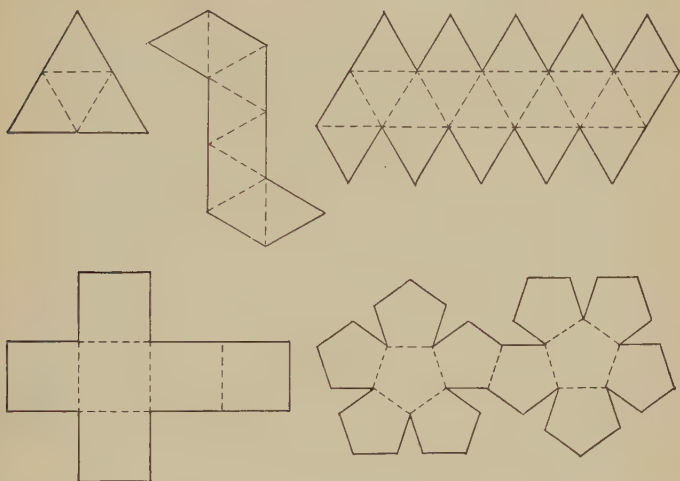
(a) Using equilateral triangles for faces, only three regular polyhedrons are possible, since polyhedral angles can be formed using three, four, or five faces. ($3 \times 60^\circ = 180^\circ$, $4 \times 60^\circ = 240^\circ$, and $5 \times 60^\circ = 300^\circ$; but $6 \times 60^\circ = 360^\circ$.)

(b) Using squares for faces, only one regular polyhedron is possible. ($3 \times 90^\circ = 270^\circ$; but $4 \times 90^\circ = 360^\circ$.)

(c) Using regular pentagons, only one regular polyhedron is possible. ($3 \times 108^\circ = 324^\circ$; but $4 \times 108^\circ > 360^\circ$.)

(d) In the case of regular polygons of more than five sides, no polyhedral angles can be formed. Why?

154. The five regular polyhedrons. Enlarging the following patterns the five regular polyhedrons may be constructed.



155. Prismatoid Formula.—The following formula is conveniently useful in practical affairs: $V = \frac{h}{6}(B + b + 4m)$,

where B and b are the areas of the bases, m the area of a section midway between the two bases, and h the perpendicular distance between the two parallel planes of the bases.

Although the formula refers to a prismatoid, or a polyhedron all of whose vertices lie in two parallel planes, it may be used to find the volumes of any sphere, cone, cylinder, pyramid, or prism, or of any frustum of one of these solids intercepted by two parallel planes.

When using this formula for the volume of a prism, show that $b = m = B$ (§§ 94, 95); hence $V = \frac{h}{6}(B + B + 4B) = Bh$.

In its application to the pyramid, show that $b = 0$ and $m = \frac{1}{4} B$ (§ 140); hence $V = \frac{h}{6}(0 + B + B) = \frac{Bh}{3}$.

Show that in a frustum of a pyramid $4m = B + b + 2\sqrt{Bb}$; hence $V = \frac{h}{6}(2B + 2b + 2\sqrt{Bb}) = \frac{h}{3}(B + b + \sqrt{Bb})$.
 $\left(\frac{e}{\frac{1}{2}(E + e)} = \frac{\sqrt{b}}{\sqrt{m}} \text{ (§§ 369; 317v); } \frac{E}{\frac{1}{2}(E + e)} = \frac{\sqrt{B}}{\sqrt{m}}; \right.$
 $\therefore \frac{E + e}{\frac{1}{2}(E + e)} = \frac{\sqrt{B} + \sqrt{b}}{\sqrt{m}} \text{ (§ 317, 1). } \therefore \frac{2}{1} = \frac{\sqrt{B} + \sqrt{b}}{\sqrt{m}}$
 and $2\sqrt{m} = \sqrt{B} + \sqrt{b}. \therefore 4m = \text{etc.} \left. \right)$

Exercises

1. Find the ratio of the volumes of two similar tetrahedrons whose corresponding edges are in the ratio 1:8. Find the ratio of the edges if the volumes are in the ratio 1:8.
2. The volumes of two similar polyhedrons are 64 cu. ft. and 216. If the area of the surface of the first is 112 sq. ft., what is the area of the second?
3. Find the volume and total surface of a regular tetrahedron each of whose edges is 6.
4. Find the volume and total surface of a regular hexahedron each of whose edges is 6.
5. Find the volume and total surface of a regular octahedron each of whose edges is 6.
6. Find the ratio of the volumes of two similar polyhedrons whose corresponding edges are in the ratio 8:27. Find the ratio of the edges if the volumes are in the ratio 8:27.
7. Using the prismatoid formula (§ 155), find the volume of a regular hexagonal prism, if its altitude is 9 and each edge of its base is 4. Find the volume of a regular hexagonal pyramid, if the altitude is 9 and each edge of the base is 4.

BOOK THREE

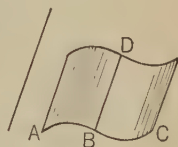
CYLINDERS AND CONES

Cylinders

156. A **curved surface** is a surface no part of which is plane.

157. A **cylindrical surface** is a curved surface generated by a moving straight line that constantly touches a plane curve and remains parallel to a given straight line not in the plane of the curve.

An *element* of a cylindrical surface is the *generator*, or moving line, in any one position.

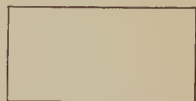


158. A **cylinder** is a solid bounded by a closed cylindrical surface and two parallel planes.

The *bases* of a cylinder are the two parallel plane surfaces; the *altitude* is the perpendicular drawn between the planes of the bases; the *lateral surface* is the cylindrical surface included between the bases.

159. A **right cylinder** is a cylinder whose elements are perpendicular to the planes of the bases. In an *oblique cylinder* the elements are not perpendicular to the bases.

The lateral surface of a right cylinder may be spread out in the form of a rectangle whose



Lateral Area

altitude is an element of the cylinder and whose base is the circumference of the base of the cylinder.

160. A **circular cylinder** is a cylinder whose bases are circles.

Since the circle is the only curve studied in elementary geometry we shall limit our study to circular cylinders.

161. A **plane is tangent to a cylinder** if it contains an element, but no other point of the cylinder.

Theorems Proved Informally

162. Any two elements of a cylinder are equal and parallel (§§ 158; 157; 13).

163. A line drawn through any point in the rim of the base of a cylinder parallel to an element is an element.

164. A right circular cylinder may be generated by the revolution of a rectangle about one of its sides as an axis (§§ 24; 159; 160).

A right circular cylinder may be called a *cylinder of revolution*.

Exercises

1. What is the locus of points at a given distance from a given unlimited straight line?

2. Given a plane and a line perpendicular to it. Find the locus of all points two units from the plane and at the same time two units from the line.

3. Find the locus of points in space at a given distance from a given line and also equidistant from two given points on that line.

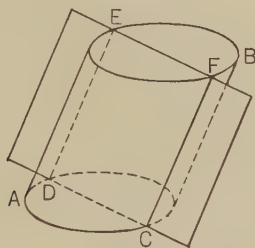
4. Are all circular cylinders necessarily cylinders of revolution?

5. Are the right sections of all circular cylinders circles?

6. Is an element of a circular cylinder necessarily an altitude? Is an element of a cylinder of revolution an altitude?

Proposition I. Theorem

165. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



Given: The cylinder AB with the section $CDEF$ made by a plane passing through the element CF .

To prove: $CDEF$ is a $\square$.

Plan of Attack: class — a figure is a parallelogram.

method to be used — show that the lines of intersection form a quadrilateral with its opposite sides parallel.

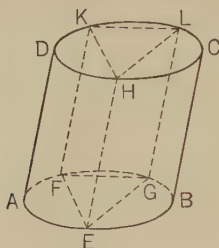
Proof:

- | | |
|---|--------------|
| 1. $FE \parallel CD$. | 1. Why? |
| 2. Through D in the rim of the base draw in the cutting plane a line $\parallel CF$. | 2. § 313. |
| 3. This line is an element, hence lies in the cylindrical surface. | 3. § 163. |
| 4. $\therefore$ it is the intersection of the plane CE with the cylindrical surface and coincides with DE . | 4. § 9b. |
| 5. $\therefore DE \parallel CF$. | 5. Why? |
| 6. $\therefore CDEF$ is a $\square$. | 6. § 339, a. |

166. Corollary. — Every section of a right cylinder made by a plane perpendicular to the base is a rectangle.

Proposition II. Theorem

167. *The bases of a cylinder are congruent.*



Given: The cylinder AC with bases AB and DC .

To prove: Base AB = base DC .

Plan of Attack: class — congruent figures.

method to be used — show that they may be made to coincide.

Proof:

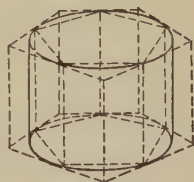
- | | |
|---|-----------------|
| 1. Through <i>any</i> three points E , F , and G in the rim of the base, draw the elements EH , FK , and GL . | 1. § 313. |
| 2. Draw EF , FG , GE , HK , KL , and LH . | 2. § 307. |
| 3. $EH = FK$ and $EH \parallel FK$. | 3. § 162. |
| 4. $\therefore EFKH$ is a $\square$. | 4. Why? |
| 5. $\therefore EF = HK$. | 5. Why? |
| 6. In a similar manner $FG = KL$ and $GE = LH$. | 6. Reasons 3-5. |
| 7. $\therefore \triangle EFG = \triangle HKL$. | 7. § 320c. |
| 8. $\therefore E$, F , and G , <i>any three</i> points in the rim of the base, may be made to coincide with the corresponding points H , K , and L in the rim of the upper base. | 8. Why? |
| 9. $\therefore$ the base AB = the base DC . | 9. § 389. |

168. Corollary I. — The sections of a cylinder made by parallel planes cutting all the elements are congruent.

169. Corollary II. — All sections of a circular cylinder parallel to the bases are equal circles, and the straight line joining the centers of the bases passes through the centers of all the parallel sections.

Measurement of Cylinders

170. A curved surface cannot be measured in terms of a plane surface unit. We can assume, however, that the surface of a cylinder has a definite area which lies between the surface areas of the inscribed and of the circumscribed regular prisms.



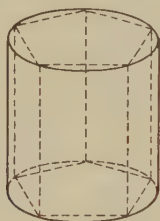
If a regular prism is inscribed in a cylinder (*i.e.* its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder), its lateral area may be measured. Likewise the lateral area of a circumscribed prism (*i.e.* one with its lateral faces tangent to the cylinder and its bases circumscribed about the bases of the cylinder) may be measured.

If the number of lateral faces of the inscribed and of the circumscribed prisms be repeatedly doubled, we notice that the difference between the surfaces of the inscribed and of the circumscribed prisms becomes less and less; and can be made less than any assigned value. We may say, therefore, that the surface areas of prisms inscribed in and circumscribed about a cylinder approach a common limiting value called the *area of the cylinder*.

171. The Cylinder as a Limit. — The lateral area of a cylinder is the limit which the lateral areas of regular inscribed and circumscribed prisms approach when the number of their lateral faces is indefinitely increased; the volume of the cylinder is the limit of the volumes of these prisms; and the base of the cylinder is the limit of the bases of the prisms.

Proposition III. Theorem

172. *The lateral area of a right circular cylinder is equal to the product of the circumference of its base and its altitude.*



Given: A right circular cylinder with its lateral area denoted by S , the circumference of its base by c , and its altitude by h .

To prove: $S = ch$.

Plan of Attack: class — areas of figures.

method to be used — informal proof by inference, comparing the area of the cylinder with the area of a regular inscribed prism.

Proof:

- | | |
|---|--|
| <p>1. Inscribe within (or circumscribe about) the cylinder a regular prism. Denote its lateral area by S', the perimeter of its base by p, and its altitude by h.</p> <p>2. Then $S' = ph$.</p> <p>3. If the number of sides of the base of the inscribed prism be increased by successive doubling, the lateral surface of each successive prism becomes more nearly equal to the lateral surface of the cylinder, and the perimeter of the base of each prism more nearly equal to the circumference of the base of the cylinder.</p> | <p>1. § 170.</p> <p>2. Why?</p> <p>3. § 170.</p> |
|---|--|

4. Since this process may be continued, the lateral area and the perimeter of the base of the prism can be made approximately equal to the lateral area and the circumference of the base of the cylinder.

5. For each successive prism $S' = ph$.

6. It is reasonable, therefore, to conclude that $S = ch$.

4. § 171.

5. Why?

6. By inference.

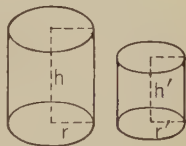
Note: Since $C = 2\pi r$, $S = 2\pi rh$.

173. Corollary. — The total area of a right circular cylinder equals the sum of its lateral area and the areas of its bases.

$$S = ch = 2\pi rh. \quad \therefore T = 2\pi rh + 2\pi r^2 = 2\pi r(h + r).$$

174. Similar cylinders of revolution are cylinders generated by revolving similar rectangles about corresponding sides as axes.

175. Corollary. — The lateral areas (or total areas) of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.



$$S = 2\pi rh \text{ and } S' = 2\pi r'h' \text{ (§ 172).}$$

$$\therefore \frac{S}{S'} = \frac{2\pi rh}{2\pi r'h'} = \frac{rh}{r'h'} \text{ (Ax. IV).}$$

$$\text{But } \frac{r}{r'} = \frac{h}{h'} \text{ (§ 398).}$$

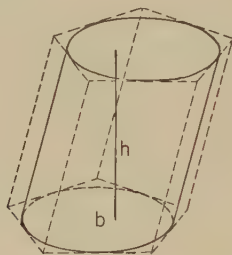
$$\therefore \frac{rh}{r'h'} = \frac{h^2}{h'^2} \text{ and } \frac{r^2}{r'^2} = \frac{rh}{r'h'} \text{ (Ax. III).}$$

$$\therefore \frac{S}{S'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2} \text{ (Ax. VIII).}$$

$$\frac{T}{T'} = \frac{2\pi r(h + r)}{2\pi r'(h' + r')} = \frac{r(h + r)}{r'(h' + r')}, \text{ but } \frac{h + r}{h' + r'} = \frac{r}{r'} \text{ (§ 363). } \therefore \text{etc.}$$

Proposition IV. Theorem

176. *The volume of a circular cylinder is equal to the product of its base and its altitude.*



Given: A circular cylinder with its volume denoted by V , the area of its base by b , and its altitude by h .

To prove: $V = bh$.

Plan of Attack: class — volumes of solids.

method to be used — informal proof by inference, comparing the volume of the cylinder with the volume of a circumscribed prism.

Proof:

- | | |
|---|---|
| <p>1. Circumscribe about the cylinder a regular prism. Denote its volume by V', the area of its base by b', and its altitude by h.</p> <p>2. Then $V' = b'h$.</p> <p>3. If the number of sides of the base of the circumscribed prism be increased by successive doubling, the volume of each successive prism becomes more nearly equal to the volume of the cylinder, and the area of the base of each prism more nearly equal to the area of the base of the cylinder,</p> <p>4. Since this process may be continued, the volume and the area of the base of the prism</p> | <p>1. § 170.</p> <p>2. Why?</p> <p>3. §§ 170; 351.</p> <p>4. § 171.</p> |
|---|---|

can be made approximately equal to the volume and the area of the base of the cylinder.

5. For each successive prism $V' = b'h$.

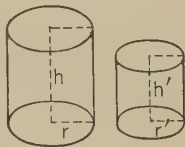
5. Why?

6. It is reasonable, therefore, to conclude that $V = bh$.

6. By inference.

Note: Since $b = \pi r^2$, $V = \pi r^2 h$.

177. Corollary. — The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.



$$V = \pi r^2 h \text{ and } V' = \pi r'^2 h' \text{ (§ 176).}$$

$$\therefore \frac{V}{V'} = \frac{\pi r^2 h}{\pi r'^2 h'} = \frac{r^2 h}{r'^2 h'} \text{ (Ax. IV).}$$

But $\frac{r}{r'} = \frac{h}{h'}$ (Why?) $\therefore \frac{r^3}{r'^3} = \frac{r^2 h}{r'^2 h'}$ (Ax. III).

Also $\frac{r^2}{r'^2} = \frac{h^2}{h'^2}$ (Ax. V). $\therefore \frac{r^2 h}{r'^2 h'} = \frac{h^3}{h'^3}$ (Ax. III).

$$\therefore \frac{V}{V'} = \frac{h^3}{h'^3} = \frac{r^3}{r'^3} \text{ (Ax. VIII).}$$

Exercises

1. Different cylinders are generated by the revolution of a rectangle about one of its sides, according as it revolves about the longer or the shorter side. Show that the curved surfaces generated in the two cases are equal.

2. How must the dimensions of a cylinder be increased in order to form a similar cylinder whose total surface shall be 9 times that of the original cylinder?

3. A solid has the form of a regular hexagonal prism of base edge e and of height h , through which a circular cylindric hole of diameter e has been cut perpendicularly from base to base. Find the total surface of the resulting solid.

4. Find the area of the lateral surface of a hot-water tank 60 inches high, which is 12 inches in diameter.

5. The lateral area of a right circular cylinder whose radius is 4 feet is one half of the total area. Find the lateral area of a similar cylinder whose radius is 8 feet.

6. Find the total area and the volume of the cylinder generated by revolving a rectangle $4'' \times 6''$ about its longer side as an axis.

7. Find the volume and surface of the solid generated by a door 3 feet wide and 8 feet high, swinging in an arc of 144° .

8. The edge of a cube is 4 inches. Find the altitude of a right circular cylinder of equal volume whose base is inscribed in the base of the cube.

9. Prove algebraically that the volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

10. Find the weight of 5 miles of copper wire $\frac{5}{16}$ of an inch in diameter. (1 mile = 5280 feet; 1 cu. ft. of copper weighs 556 lb.)

11. Two tanks are in form similar cylinders; one holds 128 gallons, the other 250 gallons. If the first is 40 inches deep, find the depth of the second.

12. A cylindric tank 6 feet long and 3 feet in diameter, lying with its axis horizontal, contains gasoline to the depth of 9 inches at the middle of the cross section. How many gallons of gasoline are in the tank?

13. When a solid is placed under water in a right circular cylinder 12 inches in diameter, the level of the water rises 3 inches. Find the volume of the solid.

14. If you were to cut off a piece of lead pipe $2''$ in outside diameter and $\frac{1}{4}''$ thick, so that it would melt into a cube whose edges are $6''$, how long a piece would be required?

15. The diameter of a cylindric hot-water tank that contains 30 gallons is 12 inches. Allowing 231 cubic inches to a gallon, find the height of the tank.

Cones

178. A **conical surface** is a surface generated by a moving straight line which constantly touches a given fixed curve and constantly passes through a given fixed point not in the plane of the curve.

An *element* of a conical surface is the *generator*, or moving line, in any one position; the *directrix* is the fixed curve; and the *vertex* is the fixed point.

It is evident that a conical surface has two parts, one on each side of the vertex; these are called *nappes*.



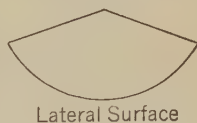
179. A **cone** is a solid bounded by a conical surface and a plane cutting all its elements.

The *base* of a cone is the part of its surface formed by the plane; the *lateral surface* is the curved part of its surface; the *altitude* is the perpendicular from the vertex to the base.

180. A **circular cone** is a cone whose base is a circle.

The *axis* of a circular cone is a line from the vertex to the center of the base. We shall study only circular cones.

181. A **right circular cone** is a circular cone whose axis is perpendicular to its base. In an *oblique* circular cone the axis is not perpendicular to the base. The lateral surface of a right circular cone may be spread out in the form of a sector of a circle whose radius is an element of the cone and whose arc is the circumference of the base of the cone.



Lateral Surface

182. The **slant height** of a right circular cone is any one of its elements.

Theorems Proved Informally

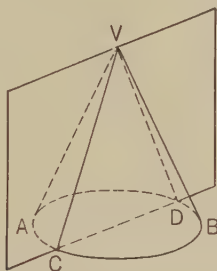
- 183.** The axis of a right circular cone is an altitude.
- 184.** The elements of a right circular cone are equal.
- 185.** A straight line from the vertex of a cone to any point in the rim of the base is an element.
- 186.** A right circular cone may be generated by the revolution of a right triangle about one of its legs as an axis (§§ 24, 181).
- A right circular cone may be called a *cone of revolution*.

Exercises

1. A cylindric tank having an internal diameter of 12 feet is filled with water to a height of 8 feet. What is the length of a six-inch pipe which this water will exactly fill?
2. A solid has the form of a regular hexagonal prism of base edge e and of height h , through which a circular cylindric hole of diameter e has been cut perpendicularly from base to base. Find the volume of the resulting solid.
3. An irregular piece of stone placed in a cylinder partly filled with water causes the water to rise three inches. If the diameter of the cylinder is 4 inches, what is the volume of the stone?
4. A gasoline tank of an automobile is a cylinder 35'' long and 15'' in diameter. How many gallons will it hold? (1 gal. = 231 cu. in.)
5. The cross section of a tunnel one mile in length is in the form of a rectangle 6 yards wide and 4 yards high, surmounted by a semi-circle whose diameter is equal to the width of the rectangle. How many cubic yards of material were taken out in the construction of the tunnel? (1 mile = 1760 yards. Use $\pi = 3.1416$.)

Proposition V. Theorem

187. *Every section of a cone made by a plane passing through its vertex is a triangle.*



Given: The cone $V-ACB$ with the section VCD formed by a plane passing through the vertex V .

To prove: VCD is a triangle.

Plan of Attack: class — triangles.

method to be used — show that the lines of intersection form a plane figure having three sides.

Proof:

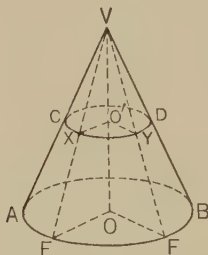
- | | |
|---|--------------------------|
| 1. The plane through V cuts the base in a straight line CD . | 1. § 10. |
| 2. Draw the straight lines CV and DV . | 2. § 307. |
| 3. CV and DV lie in the plane and are also elements in the conical surface. | 3. §§ 11 <i>b</i> ; 185. |
| 4. $\therefore CV$ and DV are the lines of intersection of the plane and the conical surface. | 4. § 9 <i>b</i> . |
| 5. $\therefore$ the section VCD is a $\triangle$. | 5. By definition. |

Exercise

A straight line intersects a plane at right angles. Find the locus of points in space equally distant from the line and the plane.

Proposition VI. Theorem

188. *In a circular cone, a section made by a plane parallel to the base is a circle.*



Given: A circular cone VAB with section CD made by a plane parallel to the base AB .

To prove: CD is a circle.

Plan of Attack: class — a figure is a circle.

method to be used — show that all points in the rim of the section are equidistant from a point in the plane of the section.

Proof:

- | | |
|---|-----------------|
| 1. Draw the axis VO cutting the section in O' . | 1. § 180. |
| 2. Through VO and <i>any</i> two points X and Y in the rim of the section pass planes cutting the base in OE and OF , respectively. | 2. §§ 4; 10. |
| 3. Then $XO' \parallel ?$ and $YO' \parallel ?$ | 3. Why? |
| 4. $\therefore \triangle VO'X \sim \triangle ?$ and $\triangle VO'Y \sim \triangle ?$ | 4. § 362. |
| 5. $\therefore \frac{XO'}{EO} = ?$ and $\frac{YO'}{FO} = ?$ | 5. Why? |
| 6. $\therefore \frac{XO'}{EO} = \frac{YO'}{FO}$ | 6. § 317, VII. |
| 7. But $EO = ?$ | 7. Why? |
| 8. $\therefore XO' = YO'$ and CD is a circle. | 8. §§ 359; 395. |

Question: When is a section of a right circular cone an ellipse; a parabola?

189. Corollary I. — The axis of a circular cone passes through the center of a section parallel to the base.

190. Corollary II. — Any section of a circular cone parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the cone.

Measurement of Cones

191. Relation to Pyramids. — Just as the theorems concerning the lateral area and volume of the cylinder are developed from the corresponding theorems for the prism, so we may derive the properties of cones from those of pyramids. In order to do so we may assume the following statements.

192. The Cone as a Limit. — The lateral area of a cone is the limit which the lateral areas of regular inscribed and circumscribed pyramids approach when the number of their lateral faces is indefinitely increased; and the volume of the cone is the limit of the volumes of these pyramids.

193. A plane tangent to a cone is a plane which contains an element, but no other point of the cone.

194. A pyramid is inscribed in a cone if its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

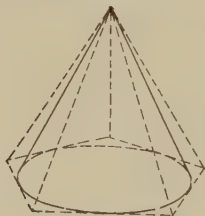
195. A pyramid is circumscribed about a cone if its lateral faces are tangent to the cone and its base is circumscribed about the base of the cone.

196. A frustum of a cone is the portion of the cone included between its base and a plane parallel to the base.

The *slant height* of a frustum of a right circular cone is that part of the slant height of the cone which is included between the base and the parallel plane.

Proposition VII. Theorem

197. *The lateral area of a cone of revolution is equal to half the product of its slant height and the circumference of its base.*



Given: A cone of revolution with the lateral area denoted by S , the element of the cone by l , and the circumference of the base by c .

To prove: $S = \frac{1}{2} cl$.

Plan of Attack: class — area of surfaces.

method to be used — informal proof by inference, comparing the area of the cone with the area of a circumscribed pyramid.

Proof:

- | | |
|---|---|
| <p>1. Circumscribe about the cone a regular pyramid. Denote its lateral area by S', the perimeter of the base by p, and its slant height by l.</p> <p>2. Then $S' = \frac{1}{2} pl$.</p> <p>3. If the number of sides of the base of the circumscribed pyramid be increased by successive doubling, the lateral surface of each successive pyramid becomes more nearly equal to the lateral surface of the cone, and the perimeter of the base of each pyramid more nearly equal to the circumference of the base of the cone.</p> <p>4. Since this process may be continued, the lateral area and the perimeter of the base of</p> | <p>1. § 195.</p> <p>2. Why?</p> <p>3. § 351; by inference.</p> <p>4. § 192.</p> |
|---|---|

the pyramid can be made approximately equal to the lateral area and the circumference of the base of the cone.

5. For each successive pyramid $S' = \frac{1}{2} pl$.

6. It is reasonable, therefore, to conclude that $S = \frac{1}{2} cl$.

5. Why?

6. By inference.

Note: Since $c = 2\pi r$, $S = \frac{1}{2}(2\pi r)\frac{1}{2}l = \pi rl$.

198. Corollary I. — The lateral area of a cone of revolution is equal to the product of its slant height by the circumference of the circle midway between the base and the vertex.

199. Corollary II. — The total area of a cone of revolution equals the sum of its lateral area and the area of its base.

$$T = \pi rl + \pi r^2 = \pi r(l + r).$$

200. Corollary III. — The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of the bases multiplied by the slant height of the frustum; or to the circumference of the midsection multiplied by the slant height.

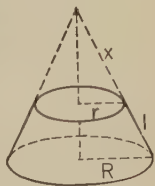
S = lateral area of the completed cone
minus the lateral area of the
small cone.

$$\therefore S = \pi R(l + x) - \pi rx \text{ (§ 197)} = \pi Rl + \pi x(R - r).$$

$$\text{But } \frac{x}{x + l} = \frac{r}{R} \text{ (§§ 12; 362; 398). } \therefore x = \frac{rl}{R - r}.$$

$$\therefore S = \pi Rl + \pi rl \text{ (Ax. 8)} = \pi l(R + r).$$

If a denotes the radius of the midsection, $a = \frac{1}{2}(R + r)$ (§ 380 c). $\therefore 2\pi al = 2\pi l \cdot \frac{1}{2}(R + r) = \pi l(R + r)$ and $S = 2\pi al$ (Ax. VIII).



201. Similar cones of revolution are cones generated by revolving similar right triangles about corresponding sides as axes.

202. Corollary IV. — The lateral areas (or total areas) of similar cones of revolution are to each other as the squares of any two corresponding lines.

$$S = \pi r l \text{ and } S' = \pi r' l' \text{ (§ 197).}$$

$$\therefore \frac{S}{S'} = \frac{\pi r l}{\pi r' l'} = \frac{r l}{r' l'} \text{ (Ax. IV).}$$

$$\text{But } \frac{r}{r'} = \frac{l}{l'} = \frac{h}{h'} \text{ (§ 201).}$$

$$\therefore \frac{r l}{r' l'} = \frac{l^2}{l'^2} \text{ (Ax. III) and } \frac{r^2}{r'^2} = \frac{l^2}{l'^2} = \frac{h^2}{h'^2} \text{ (Ax. V).}$$

$$\therefore \frac{S}{S'} = \frac{l^2}{l'^2} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2} \text{ (Ax. VIII).}$$

$$\frac{T}{T'} = \frac{\pi r(l+r)}{\pi r'(l'+r')} = \frac{r(l+r)}{r'(l'+r')}; \text{ but } \frac{l+r}{l'+r'} = \frac{r}{r'} \text{ (§ 363); etc.}$$



Exercises

1. Find the lateral surface of a cone of revolution circumscribing a regular hexagonal pyramid each edge of whose base is 8 inches and whose altitude is 15 inches.

2. Find the total area of a cone inscribed in a regular triangular pyramid whose altitude is 12 inches and whose lateral edge is 15 inches.

3. The altitude of the frustum of a cone is 12 inches; the radii of the bases are respectively 5 inches and 14 inches. Find the total area of the frustum.

4. The sides of a parallelogram, which are 12 inches and 8 inches respectively, form an angle of 60° . Find the surface generated by revolving the parallelogram about one of its longer sides as an axis.

5. At two cents per square foot, what will it cost to paint the inside of an open cistern having the form of a frustum of a cone of revolution whose depth is 8 feet, whose top diameter is 4 feet, and bottom diameter 8 feet?

6. A regular hexagon whose side is R revolves about a longest diagonal as an axis. Find in terms of R the entire surface generated.

7. If the slant height of a cone of revolution is $18''$, find the parts into which the slant height is divided by a plane which is parallel to the base and which divides the convex surface into parts having the ratio $4:5$.

8. A silver dish in the form of a conical frustum is to be plated inside with gold; the upper diameter of the dish is 8 inches, the diameter of the base is 6 inches, and the height of the dish is 3 inches. How many square inches of gold plating will be required?

9. A right triangle whose legs are 6 inches and 8 inches in length revolves about the hypotenuse as an axis. Find the area of the resulting solid.

10. The sides of a triangle are 8, 17, and 21. Find the surface generated by revolving the triangle about the side 21 as an axis.

11. A right cylinder and a right cone 12 inches in height stand on the same circular base whose radius is 9 inches. The altitude of the cylinder is 4 inches. Find the lateral surface of that part of the cone which is above the cylinder.

12. A plane parallel to the base of a right circular cone cuts the altitude at a point $\frac{2}{5}$ of the distance from the vertex to the base. Find the ratio of the lateral areas of the two solids formed.

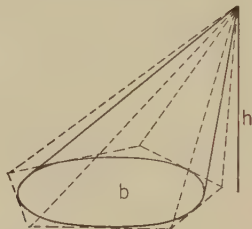
13. A round tower has a roof in the form of a frustum of a cone, the sloping sides of which make an angle of 45° with the horizontal. The diameter at the bottom is 31.5 ft. and at the top 15.5 ft. Find the number of square feet in the roof of the tower.

14. Find the weight of a cast-iron water pipe 12 feet long, outside diameter 2 inches, thickness of the iron $\frac{1}{4}$ of an inch, if the iron weighs 0.26 lb. per cubic inch.

15. Show that the volume of a cylinder may be found by using the prismatoid formula, § 155.

Proposition VIII. Theorem

203. *The volume of a circular cone is equal to one third the product of its base and its altitude.*



Given: A circular cone with its volume denoted by V , the area of its base by b , and its altitude by h .

To prove: $V = \frac{1}{3}bh$.

Plan of Attack: class — volumes of solids.

method to be used — informal proof by inference, comparing the volume of the cone with the volume of a circumscribed pyramid having a regular base.

Proof:

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| <p>1. Circumscribe about the cone a pyramid. Denote its volume by V', the area of its base by b', and its altitude by h.</p> <p>2. Then $V' = \frac{1}{3}bh$.</p> <p>3. If the number of sides of the base of the circumscribed pyramid be increased by successive doubling, the volume of each successive pyramid becomes more nearly equal to the volume of the cone, and the area of its base more nearly equal to the area of the base of the cone.</p> <p>4. Since this process may be continued, the volume and the area of the base of the pyra-</p> | <p>1. § 195.</p> <p>2. Why?</p> <p>3. § 351; by inference.</p> <p>4. § 192</p> |
|---|--|

mid can be made approximately equal to the volume and the area of the base of the cone.

- | | |
|---|------------------|
| 5. For each successive pyramid $V' = \frac{1}{3} b'h$. | 5. Why? |
| 6. It is reasonable, therefore, to conclude that $V = \frac{1}{3} bh$. | 6. By inference. |

Note: Since $b = \pi r^2$, $V = \frac{1}{3} \pi r^2 h$.

204. Corollary I. — The volumes of two similar cones of revolution are to each other as the cubes of any two corresponding lines.

$$V = \frac{1}{3} \pi r^2 h \text{ and } V' = \frac{1}{3} \pi r'^2 h' \text{ (§ 203).}$$

$$\therefore \frac{V}{V'} = \frac{\frac{1}{3} \pi r^2 h}{\frac{1}{3} \pi r'^2 h'} = \frac{r^2 h}{r'^2 h'} \text{ (Ax. IV).}$$

But $\frac{r}{r'} = \frac{h}{h'} \text{ (§ 398).} \quad \therefore \frac{r^3}{r'^3} = \frac{r^2 h}{r'^2 h'} \text{ (Ax. III).}$

Also $\frac{r^3}{r'^3} = \frac{h^3}{h'^3} = \frac{l^3}{l'^3} \text{ (Ax. V).}$

$$\therefore \frac{V}{V'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3} = \frac{l^3}{l'^3} \text{ (Ax. VIII).}$$

205. Corollary II. — The volume of a frustum of a cone of revolution is expressed by the formula, $V = \frac{1}{3} h(\pi R^2 + \pi r^2 + \pi rR)$, where R and r are the radii of the bases and h is the altitude. (Prove in a manner similar to § 147.)

Exercises

1. Find the volume of a cone of revolution circumscribing a regular hexagonal pyramid each edge of whose base is 8 inches and whose altitude is 15 inches.

2. The hypotenuse of a right triangle is 15 inches, one leg is 9 inches. Find the volume of the solid generated by revolving the triangle on its hypotenuse as an axis.

3. A triangle whose sides are 15, 13, and 4, revolves about its shortest side as an axis. Find the volume of the solid generated by the revolving triangle.

4. The sides of a parallelogram which are 12 and 8 form an angle of 60° . Find the volume of the solid generated by revolving the parallelogram about one of its longer sides.

5. The volume of the frustum of a cone of revolution is 105π cubic inches, the altitude of the frustum is 5 inches, and the radius of the upper base is 3 inches. Find the radius of the lower base.

6. In a triangle the sides including an angle of 120° are s and $2s$. Find the volume of the solid generated by revolving the triangle about the shortest side as an axis.

7. A tank in the form of a frustum of a right cone is 18 feet in diameter at the bottom, 16 feet at the top, and 14 feet deep. Find its capacity in cubic feet.

8. The diameter of the base of a right circular cone is 10 feet, its altitude is 18 feet. Find the altitude of a right circular cylinder of equivalent volume, the diameter of whose base is 14 feet.

9. A right triangle whose altitude is 4 inches and whose area is 6 square inches is revolved about its other leg as an axis. Find the volume of the solid generated.

10. An isosceles trapezoid, with bases 4 and 6 and altitude 3, is revolved about its longer base. Find the volume of the solid generated.

11. A cone 5 inches high is cut by a plane parallel to the base and 2 inches from the base; the volume of the frustum thus formed is 294 cubic inches. Find the volume of the cone and the volume of the part cut off by the plane.

12. The sides of a triangle are each 12 inches. Find (a) the surface and (b) the volume of the solid generated by revolving the triangle about one side.

13. Find the altitude of a frustum of a circular cone, if its volume equals 190 cu. in. and the radii of its bases are 2 in. and 3 in.

14. In each of two right circular cones the diameter of the base is equal to the height. What is the ratio of the heights if the volumes of the cones are as 16 : 27 ?

15. From a right circular frustum, whose upper base diameter is $2a$ and whose lower diameter is $4a$, is bored out a right circular cone whose base lying in the lower base of the frustum has a diameter of $2a$ and whose vertex is in the plane of the upper base of the frustum. Find the ratio of the volume of the original frustum to the volume of the hollow frustum.

16. How many bricks are necessary to build a chimney 80 feet high, in the shape of a frustum of a cone, if the outer diameters are 3 feet and 7 feet and the inner diameters are 2 feet and 4 feet ? (Allow 12 bricks to a cubic foot.)

17. A cone is inscribed in a regular triangular pyramid of altitude h inches and base b square inches. Find the number of cubic inches between the surfaces of the two solids.

18. A right triangle whose legs are 6 and 8 revolves about the hypotenuse as an axis. Find the volume of the resulting solid.

19. The sides of a triangle are 8, 17, and 21. Find the volume of the solid generated by revolving the triangle about the side 21.

20. A water pail is in the shape of a frustum of a cone, the diameters of the bottom and top being 9'' and 12'' respectively, and the height of the pail 14''. How many quarts does it hold ? (One gallon contains 231 cubic inches.)

21. Given a cone of revolution of altitude h , find in terms of h how far from the vertex two planes must be passed parallel to the base to divide the cone into three equivalent parts.

22. The volumes of two similar cones of revolution are 343 and 512 respectively. If the lateral area of the first is 196, what is the lateral area of the second ?

23. A plane parallel to the base of a right circular cone cuts the altitude at a point $\frac{2}{3}$ of the distance from the vertex to the base. Find the ratio of the volumes of the two solids formed.

24. The top and bottom diameters of a tub are 30'' and 24'' and its depth is 10''. How many gallons of water are there in the tub when it is filled to a depth of 8''?

25. An oblique cone with a circular base has an altitude of 8 feet. Its shortest and longest elements are 10 feet and 17 feet. Find the volume of the cone.

26. Half of a regular hexagon inscribed in a semicircle whose radius is 6 is revolved about the diameter of the semicircle as an axis. Find the surface and volume generated by the semipolygon.

27. An open cistern in the form of a frustum of a cone of revolution, whose depth is 7 feet, whose top diameter is 3 feet, and bottom diameter 6 feet, is filled with water. How many gallons of water are in the cistern, if a cubic foot of water contains $7\frac{1}{2}$ gallons?

28. The slant height of a regular hexagonal pyramid is 8 and a side of the base 6. Find the volume of the inscribed cone.

29. A semicircle of cardboard with diameter 12'' is rolled into a cone. Find the lateral surface and volume of the cone.

30. Each edge of a regular tetrahedron is 6''. Find the lateral area and the volume of the circumscribed cone.

31. A piece of paper in the shape of a sector of a circle is rolled into a cone. Find the lateral area and the volume of the cone, if the radius of the sector is 10 and the angle of the sector is 288° .

32. Using § 190, show that the volume of a cone may be found by using the prismatoid formula § 155.

33. Show that the volume of a frustum of a cone of revolution may be found by using the prismatoid formula.

34. A right circular cone has the radius of the base equal to 4 and the altitude equal to 12. Find the lateral area and volume of the cone. Find the lateral area and the volume of the inscribed regular quadrangular pyramid.

BOOK FOUR

THE SPHERE

206. A **sphere** is a closed surface all points of which are equidistant from a fixed point called the *center*.

A sphere may be generated by revolving a semicircle about its diameter as an axis.

207. A **radius** is a line from the center to any point of the sphere.

208. A **diameter** of a sphere is a line through the center terminating in the sphere.

209. Postulate. — A sphere may be described with any given point as a center and with any given line segment as a radius.

Theorems Proved Informally

210. All radii of a sphere, or of equal spheres, are equal.

211. Spheres having equal radii are equal.

212. All diameters of a sphere, or of equal spheres, are equal.

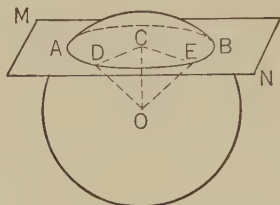
213. A point is within, on, or outside a sphere, if its distance from the center is less than, equal to, or greater than a radius.

Exercise

1. What is the locus of points at a given distance from a given point?

Proposition I. Theorem

214. *If a plane intersects a sphere, the intersection is a circle.*



Given: The sphere O intersected by the plane MN in AB .

To prove: AB is a circle.

Plan of Attack: class — a figure is a circle.

method to be used — show that all points in the intersection are equidistant from a point in the plane.

Proof:

- | | |
|---|---------------------|
| 1. Draw from the center O a line | 1. § 26. |
| $OC \perp MN$ at C . | |
| 2. Draw lines from C in MN to any | 2. § 307. |
| two points D and E in the intersection. | |
| Draw radii OD and OE . | |
| 3. $OC \perp ?$ and $OC \perp ?$ | 3. § 9c. |
| 4. $\therefore \triangle OCD$ and OCE are right $\triangle$. | 4. Why? |
| 5. $OD = ?$ and $OC = ?$ | 5. § 210; Identity. |
| 6. $\therefore$ rt. $\triangle OCD =$ rt. $\triangle OCE$. | 6. § 321a. |
| 7. $\therefore DC = EC$ and C is equidistant | 7. § 322. |
| from any two points in AB . | |
| 8. $\therefore AB$ is a circle. | 8. § 395. |

215. Corollary I. — A radius, or a diameter, of a sphere perpendicular to the plane of a circle on the sphere passes through the center of the circle; and a radius, or a diameter, of a sphere drawn through the center of a small circle on the sphere is perpendicular to the plane of the circle.

216. Corollary II. — Through any three points on a sphere one, and only one, circle may be drawn.

217. Corollary III. — The intersection of two spheres is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is in that line. (Through a point in the line of intersection pass a plane perpendicular to the line of centers. Use §§ 23, 214, 215.)

218. A great circle of a sphere is a circle whose plane passes through the center of the sphere.

All other circles of a sphere are called *small circles*.

219. The axis of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.

220. The poles of a circle of a sphere are the extremities of the axis.

Theorems Proved Informally

221. The axis of a circle of a sphere passes through the center of the circle.

222. The center of a great circle of a sphere is the center of the sphere.

223. All great circles of a sphere are equal.

224. Any two great circles of a sphere bisect each other.

225. Through two given points of a sphere, not the extremities of a diameter, one, and only one, great circle can be drawn.

226. All great circles of a sphere bisect the sphere.

227. Circles of a sphere formed by parallel planes have the same axis and the same poles.

228. Circles of a sphere equidistant from the center are equal; and conversely.

229. Circles of a sphere unequally distant from the center are unequal, the one nearest the center being the greater; and conversely.

230. The spherical distance between two points of a sphere is the length of the minor arc of the great circle passing through the points.

Exercises

1. Prove that the smallest section of a sphere made by a plane passing through a given point within the sphere is that made by a plane perpendicular to the radius through the given point.

2. Find the locus of a point in space which is at a given distance from a given point and also at a given distance from a given plane.

3. Find all possible locations of a point that is equidistant from two given points in space and at a given distance from a third point.

4. Find the locus of a point on a sphere that is equidistant from two given points on the surface.

5. A square with sides 12 lies with its vertices on a sphere whose radius is 10. How far from the center of the sphere is the plane of the square?

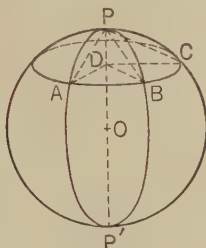
6. A point is at a given distance from a fixed point P and is also equidistant from two other fixed points A and B . What is the locus of the point?

7. Prove that if a plane cuts two concentric spheres, the sections are concentric circles.

8. A sphere of diameter 30'' is cut by a plane 9'' from the center. Find the area of a square inscribed in the circle of intersection.

Proposition II. Theorem

231. *From a given pole of a circle of a sphere, the spherical distances of all points on the circle are equal.*



Given: A , B , and C , any three points on the circle ABC ; P and P' the poles of the $\odot ABC$.

To prove: The great circle arcs PA , PB , and PC are equal.

Plan of Attack: class — equal arcs.

method to be used — on the same circle, or equal circles, equal chords have equal arcs.

Proof:

- | | |
|--|------------------|
| 1. The axis PP' passes through D , the center of the $\odot ABC$. Draw AD , BD , CD and the chords PA , PB , and PC . | 1. §§ 221 ; 308. |
| 2. $\therefore AD = ? = ?$ | 2. Why? |
| 3. $\therefore$ chord $PA =$ chord $PB =$ chord PC . | 3. § 29. |
| 4. $\therefore \widehat{PA} = \widehat{PB} = \widehat{PC}$. | 4. § 345. |
| 5. Similarly, $\widehat{P'A} = \widehat{P'B} = \widehat{P'C}$. | 5. Why? |

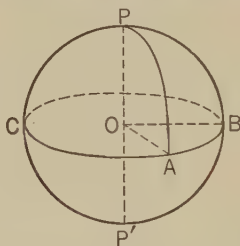
232. The **polar distance** of a circle of a sphere is the spherical distance from any point on the circle to its nearest pole.

233. A **quadrant** is one fourth of a great circle.

234. Corollary. — The *polar distance* of a great circle is a quadrant.

Proposition III. Theorem

235. *On a sphere, a point which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is the pole of the great circle through these points.*



Given: Points P , A , and B on the sphere O ; $\widehat{PA}$ and $\widehat{PB}$ are quadrants; $\widehat{AB}$ a minor arc of the great $\odot ABC$.

To prove: P is the pole of the great $\odot ABC$.

Plan of Attack: class — pole of a circle of a sphere.

method to be used — show that diameter POP' is an axis of $\odot ABC$, or that $POP' \perp$ plane ABC (§ 20).

Proof:

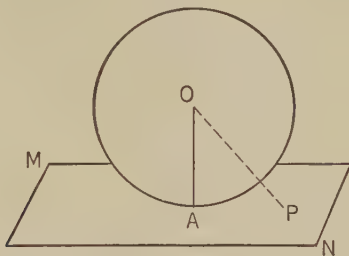
- | | |
|---|-----------|
| 1. Draw the diameter POP' ; also radii OA and OB . | 1. § 307. |
| 2. $\widehat{PA}$ and $\widehat{PB}$ are quadrants. | 2. Why? |
| 3. $\therefore \angle ?$ and $\angle ?$ are right $\angle$ s. | 3. § 348. |
| 4. $\therefore POP' \perp$ plane ABC . | 4. Why? |
| 5. $\therefore POP'$ is an axis of $\odot ABC$. | 5. § 219. |
| 6. $\therefore P$ is a pole of $\odot ABC$. | 6. § 220. |

236. A line, or a plane, is tangent to a sphere if it has one, and only one, point in common with the sphere.

237. Two spheres are tangent to each other if they are tangent to a plane at the same point.

Proposition IV. Theorem

238. *A plane perpendicular to a radius at its outer extremity is tangent to a sphere.*



Given: Sphere O with radius OA ; plane $MN \perp OA$ at A .

To prove: MN is tangent to the sphere O .

Plan of Attack: class — plane tangent to a sphere.

method to be used — show that A is the only point in MN that is on the sphere.

Proof:

1. Let P be any point in MN except A .	1. § 307.
Draw PO .	
2. $PO > OA$.	2. § 27.
3. $\therefore P$, or any point in MN except A , is outside the sphere.	3. § 213.
4. $\therefore MN$ is tangent to the sphere O .	4. § 236.

239. Corollary. — The plane tangent to a sphere at a given point is perpendicular to the radius drawn to that point.

240. A sphere is circumscribed about a polyhedron, if all the vertices of the polyhedron lie in the sphere. The polyhedron is inscribed in the sphere.

241. A sphere is inscribed in a polyhedron if every face of the polyhedron is tangent to the sphere. The polyhedron is circumscribed about the sphere.

242. Theorem. — A sphere can be inscribed in any given tetrahedron.

Hint: Bisect three dihedral angles which have their edges in one face of the tetrahedron. The point in which they meet will be the center and its distance to one of the faces will be the radius of the inscribed sphere. (§§ 71 ; 206 ; 238 ; 241.)

Exercises

1. Prove that all tangents drawn to a sphere from any external point are equal.

2. How would you locate the poles of a given great circle on a sphere?

3. Prove that if two spheres intersect, the tangent lines drawn to the spheres from any point in the plane of the circle of intersection are equal.

4. What is the locus of points at a given distance from a given line segment?

5. Find the volume and surface of a cube inscribed in a sphere whose radius is 5 inches.

6. What is the locus of the centers of spheres tangent to all three sides of a given triangle?

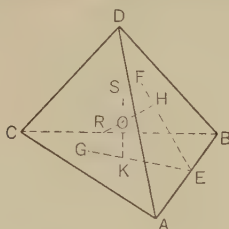
7. Find the locus of the center of a sphere which is tangent to two parallel planes and passes through a point which lies between the planes.

8. Find the locus of the center of a sphere which passes through two given points and is tangent to a given plane at a given point.

9. A right circular cone, having its altitude equal to the diameter of its base, stands with its base in one plane and its vertex in a parallel plane. A sphere is placed so that it is tangent to both planes. A plane parallel to the given planes cuts the sphere and the cone so that the sections formed are equal. How far from the lower plane is the cutting plane?

Proposition V. Theorem

243. *A sphere can be circumscribed about any tetrahedron.*



Given: Tetrahedron $D-ABC$.

To prove: A sphere may be circumscribed about $D-ABC$.

Plan of Attack: class — a point equidistant from other points.
 method to be used — intersection of loci;
 locus of points equidistant from two points,
 also locus of points equidistant from three points.

Proof:

- | | |
|--|----------------|
| 1. At E , the midpoint of AB , construct a plane $FEG \perp AB$. | 1. § 22. |
| 2. The plane $FEG \perp ABC$ and ABD . | 2. § 57. |
| 3. Also plane FEG will contain points H and K , the centers of $\odot$ circumscribed about the $\triangle ABD$ and ABC . | 3. § 70. |
| 4. In the plane FEG draw $HR \perp FE$ at H and $KS \perp GE$ at K . | 4. § 314. |
| 5. Then HR and KS will intersect in some point O . | 5. § 337. |
| 6. $HR \perp ABD$ and $KS \perp ABC$. | 6. § 59. |
| 7. $\therefore OA = OB = OD = OC$ and a sphere with center O and radius OA will pass through B, C , and D . | 7. §§ 31; 206. |

244. Corollary. — A sphere can be passed through any four points, not all of which lie in the same plane.

245. A **spherical angle** is an angle formed by two intersecting arcs of great circles.

It is equal to the angle formed by the tangents to the arcs at the point of intersection.

Exercises

1. What is the locus of points at a given distance from a given sphere?

2. Given a sphere whose radius is $2''$ and a fixed line through its center. Find the locus of points in space $3''$ from the surface of the sphere and $3''$ from the line.

3. The chord of the spherical distance from a point of a circle of a sphere to its nearest pole is 12 inches. If the radius of the sphere is 12 inches, find the area of the circle.

4. What is the locus of points 2 inches from a given point and equidistant from two other given points?

5. A and B are two points 6 inches apart. What is the locus of a point that is 4 inches from both A and B ?

6. Find, if possible, the locus of a point in space equidistant from two parallel planes $12''$ apart and also (a) $8''$ from a point in one of the planes; (b) $6''$ from a point in one of the planes; (c) $4''$ from a point in one of the planes.

7. A circle of a sphere has a radius of $3''$. If the diameter of the sphere is $10''$, how far from the center of the sphere is the plane of the circle?

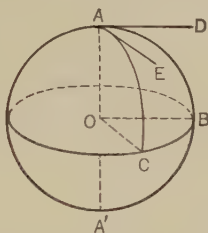
8. Two points A and B are $6''$ apart. Find the locus of points $5''$ from A and also $5''$ from B .

9. What is the locus of a point which is $3''$ from a given line and $6''$ from a fixed point in the line?

10. The diameter of a sphere is $20''$. Find the volume of the largest regular prism of altitude $16''$ which can be cut from the sphere, if the base of the prism is (a) a triangle; (b) a square; (c) a hexagon.

Proposition VI. Theorem

246. *A spherical angle is measured by the arc of a great circle described from the vertex of the angle as a pole and included between the sides of the angle, produced if necessary.*



Given: Spherical $\angle CAB$; AE and AD tangents to arcs AC and AB respectively; CB an arc of the great circle whose pole is A .

To prove: $\angle CAB = {}^\circ \text{arc } CB$.

Plan of Attack: class — measurement of angles.

method to be used — show that the spherical angle equals the central angle which is measured by the arc CB .

Proof:

- | | |
|---|------------|
| 1. Draw the diameter AOA' and the radii OB and OC . | 1. § 307. |
| 2. Arcs AC and AB are quadrants. | 2. § 234. |
| 3. $\therefore \angle ?$ and $?$ are right $\angle$ s. | 3. § 348. |
| 4. $\therefore OC \perp ?$ and $BO \perp ?$ | 4. Why? |
| 5. But $AE \perp ?$ and $AD \perp ?$ | 5. § 352. |
| 6. $\therefore OC \parallel AE$ and $OB \parallel AD$. | 6. § 336. |
| 7. $\therefore \angle COB = \angle ?$ | 7. § 42. |
| 8. $\angle COB = {}^\circ \widehat{CB}$. | 8. Why? |
| 9. $\therefore \angle EAD = {}^\circ \widehat{CB}$. | 9. Why? |
| 10. $\therefore \angle CAB = {}^\circ \widehat{CB}$. | 10. § 245. |

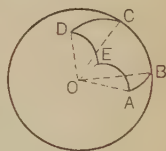
247. Corollary I. — A spherical angle is equal to the dihedral angle formed by the planes of its arcs.

248. Corollary II. — All great circles drawn through a pole of a given great circle are perpendicular to it.

Spherical Polygons and Triangles

249. A **spherical polygon** is a closed spherical figure bounded by three or more arcs of great circles ; as *ABCDE*.

The *sides* of the spherical polygon are the arcs ; the *vertices*, the points of intersection of the sides ; the *angles*, the spherical angles formed by the sides.

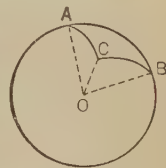


It is evident that the planes of the sides of a spherical polygon form a polyhedral angle with its vertex at the center of the sphere. The face angles of this polyhedral angle correspond to the sides of the spherical polygon, and the dihedral angles of the polyhedral angle to the angles of the spherical polygon.

250. A **spherical triangle** is a spherical polygon of three sides.

It is called isosceles, equilateral, right, etc., in the same sense as a plane triangle.

It is evident that the planes of the sides of a spherical triangle form a trihedral angle with its vertex at the center of the sphere. Hence, the trihedral angle and its parts correspond to the spherical triangle and its parts.



Before taking up the study of spherical triangles we should review §§ 75, 76, and 77.

Theorems Proved Informally

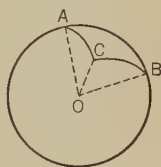
251. The sides of a spherical polygon are the measures of the face angles of the corresponding polyhedral angle at the center. Why?

252. The angles of a spherical polygon equal in measurement the corresponding dihedral angles of the polyhedral angle formed by the planes of sides of polygon (§§ 246, 55).

253. Theorem. — The sum of two sides of a spherical triangle is greater than the third side.

$$\angle AOC + \angle BOC > \angle AOB \text{ (§ 75).}$$

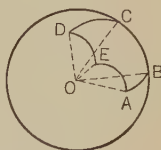
But $\angle AOC = \widehat{AC}$, $\angle BOC = \widehat{BC}$,
and $\angle AOB = \widehat{AB}$, etc.



254. Corollary. — The shortest line that can be drawn on a sphere, connecting two given points of the sphere, is an arc of a great circle.

255. Theorem. — The sum of the sides of a convex spherical polygon is less than 360° ; or less than a great circle.

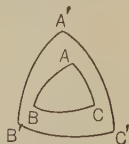
Use §§ 251 and 76.



256. The **polar triangle** of a given spherical triangle is the triangle formed by three arcs of great circles described from the vertices of the given triangle as poles.

Thus $\triangle A'B'C'$ is the polar triangle of ABC .

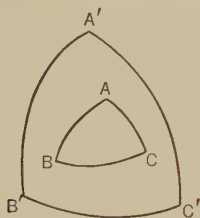
It follows that each vertex of the given triangle is a quadrant's distance from the opposite side of its polar triangle (§ 234).



Questions: Under what conditions are a triangle and its polar triangle identical? When are the sides of a polar triangle entirely within the given triangle? When is the polar triangle outside the given triangle?

Proposition VII. Theorem

257. *If one spherical triangle is the polar triangle of another, then reciprocally the second is the polar triangle of the first.*



Given: $A'B'C'$, the polar triangle of ABC ; A , B , and C being the poles of the arcs $B'C'$, $A'C'$, and $A'B'$ respectively.

To prove: ABC is the polar triangle of $A'B'C'$.

Plan of Attack: class — polar triangles.

method to be used — show that points A and B are each a quadrant's distance from C' ; hence C' is the pole of $\widehat{AB}$, etc.

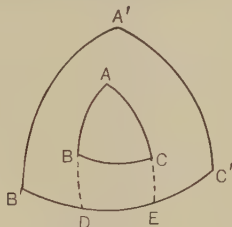
Proof:

- | | |
|--|-----------|
| 1. A and B are the poles of the arcs $B'C'$ and $A'C'$ respectively. | 1. § 256. |
| 2. $\therefore C'$ which is on the arcs $B'C'$ and $A'C'$ is a quadrant's distance from both A and B . | 2. § 234. |
| 3. $\therefore C'$ is the pole of the arc $\widehat{AB}$. | 3. § 235. |
| 4. Similarly, A' and B' are the poles of $\widehat{BC}$ and $\widehat{AC}$ respectively. | 4. Why? |
| 5. $\therefore \triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$. | 5. § 256. |

Questions: If a triangle is trirectangular, what is the relative position of the triangle and its polar triangle? If the given triangle is birectangular, what relation exists between the triangle and its polar triangle? If one side of a triangle is a quadrant, where is the polar triangle located with respect to the given triangle? (See § 262.)

Proposition VIII. Theorem

258. *In two polar triangles, any angle of either is the supplement of the opposite side of the other.*



Given: Polar $\triangle ABC$ and $A'B'C'$.

To prove: $\angle A + \widehat{B'C'} = 180^\circ$, $\angle B + \widehat{A'C'} = 180^\circ$,
 $\angle C + \widehat{A'B'} = 180^\circ$, $\angle A' + \widehat{BC} = 180^\circ$, $\angle B' + \widehat{AC} = 180^\circ$
 $\angle C' + \widehat{AB} = 180^\circ$.

Plan of Attack: class — sum equals 180° .

method to be used — show that $\widehat{B'E} + \widehat{DC'} = 180^\circ$, then use § 246 and axioms.

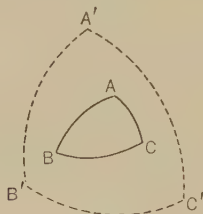
Proof:

- | | |
|--|------------------------------|
| 1. Extend arcs AB and AC to meet $\widehat{B'C'}$ at D and E respectively. | 1. § 311. |
| 2. B' is the pole of $\widehat{ACE}$ and C' is the pole of $\widehat{ABD}$. | 2. § 256. |
| 3. $\therefore \widehat{B'E} = 90^\circ$ and $\widehat{DC'} = 90^\circ$. | 3. § 234. |
| 4. $\therefore \widehat{B'E} + \widehat{DC'} = 180^\circ$ or
$\widehat{B'D} + \widehat{DE} + \widehat{DC'} = 180^\circ$. | 4. Why? |
| 5. But $\angle A = \widehat{DE}$ and
$\widehat{B'C'} = \widehat{B'D} + \widehat{DC'}$. | 5. §§ 246; 317, <i>via</i> . |
| 6. $\therefore \angle A + \widehat{B'C'} = 180^\circ$. | 6. Why? |

Similarly, each of the other angles and arcs may be proved supplementary.

Proposition IX. Theorem

259. *The sum of the angles of a spherical triangle is greater than 180° and less than 540° .*



Given: Spherical $\triangle ABC$.

To prove: $\angle A + \angle B + \angle C > 180^\circ$ and $< 540^\circ$.

Plan of Attack: class — sum of angles.

method to be used — show that $\angle A + \angle B + \angle C$ plus the sum of the sides of the polar $\triangle = 540^\circ$ and that the sum of the sides $< 360^\circ$ and $> 0^\circ$; then axiom XII.

Proof: Draw the polar $\triangle A'B'C'$, then

- | | | |
|----|--|-----------|
| 1. | $\angle A + B'C' = 180^\circ$. | 1. § 258. |
| | $\angle B + A'C' = 180^\circ$. | |
| | $\angle C + A'B' = 180^\circ$. | |
| 2. | $\therefore \angle A + \angle B + \angle C + A'B' + B'C' + A'C' = 540^\circ$. | 2. Why? |
| 3. | But $A'B' + B'C' + A'C' < 360^\circ$. | 3. § 255. |
| 4. | $\therefore \angle A + \angle B + \angle C > 180^\circ$. | 4. Why? |
| 5. | Also $A'B' + B'C' + A'C' > 0^\circ$. | 5. Why? |
| 6. | $\therefore \angle A + \angle B + \angle C < 540^\circ$. | 6. Why? |

260. Corollary. — The sum of the angles of a spherical polygon of n sides is greater than $(n - 2)180^\circ$.

261. The **spherical excess** of a spherical triangle is the number of degrees by which the sum of its angles exceeds 180° .

It represents the amount by which the sum of the angles of a spherical triangle exceeds the sum of the angles of a plane triangle. Hence, the *spherical excess* of a spherical polygon of n sides is the amount by which the sum of its angles exceeds $(n - 2)180^\circ$ (§ 356).

262. A **birectangular spherical triangle** has two right angles, and a **triectangular spherical triangle** has three right angles.

Exercises

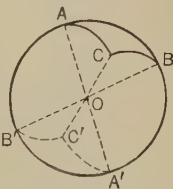
1. Prove that the exterior angle of a spheric triangle is less than the sum of the two opposite interior angles.

2. Prove that in a birectangular spheric triangle the sides opposite the right angles are quadrants and the third side measures its opposite angle.

3. Find the length of each side of a triectangular triangle.

263. **Symmetrical spherical triangles** are spherical triangles which have the sides and angles of one respectively equal to the sides and angles of the other, but arranged in reverse order.

It is evident that the planes forming two vertical trihedral angles having their common vertex at the center of a sphere form a pair of symmetrical spherical triangles on the sphere.



264. **Congruent spherical triangles** have their parts equal and arranged in the same order. They may be made to coincide.

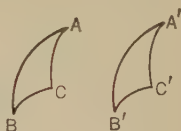
(a) Two symmetrical isosceles spherical triangles are congruent. Why?

(b) Two triangles symmetrical to the same triangle are congruent. Why?

265. Theorem. — Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetrical, if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

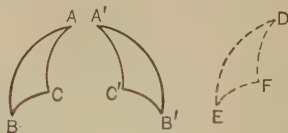
CASE I. If the equal parts occur in the same order, they are congruent.

Prove by superposition, as in plane geometry.



CASE II. If the equal parts occur in reverse order, they are symmetrical.

Let ABC and $A'B'C'$ be the given Δ in which $AB = A'B'$, $AC = A'C'$, and $\angle A = \angle A'$ arranged in reverse order.



Construct $\triangle DEF$ symmetric with $\triangle A'B'C'$. Show that $\triangle ABC$ and $\triangle DEF$ are congruent by Case I. Hence, $\triangle ABC$ and $\triangle A'B'C'$ are symmetrical.

266. Theorem. — Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetrical, if a side and the adjacent angles of one are equal respectively to a side and the adjacent angles of the other.

CASE I. If the equal parts occur in the same order, they are congruent.

Prove by superposition.

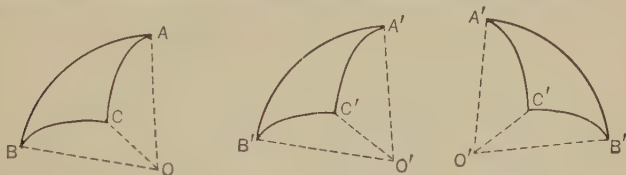
CASE II. If the equal parts occur in reverse order, they are symmetrical.

Construct a triangle symmetric with one; show that it is congruent to the other; etc.

Question: Why can we not prove case II in §§ 265 and 266 by superposition? Illustrate by cutting two triangles from the skin of an orange.

Proposition X. Theorem.

267. Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetrical, if the three sides of one are equal respectively to the three sides of the other.



Given: Spherical $\triangle ABC$ and $A'B'C'$, in which $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

To prove: $\triangle ABC$ and $A'B'C'$ are congruent or symmetrical.

Plan of Attack: class — congruent or symmetric triangles.

method to be used — show that the corresponding trihedral angles at the center are congruent or symmetric; hence the $\triangle$.

Proof:

- | | |
|--|-----------------|
| 1. From O and O' , the centers of the respective spheres, draw radii OA , OB , OC , $O'A'$, $O'B'$, and $O'C'$. | 1. § 307. |
| 2. Then the corresponding face $\angle$ of the trihedral $\angle O-ABC$ and $O'-A'B'C'$ are equal. | 2. § 251. |
| 3. $\therefore$ the corresponding dihedral $\angle$ of the trihedral $\angle O-ABC$ and $O'-A'B'C'$ are equal. | 3. § 77. |
| 4. $\therefore$ the $\angle$ of the spherical $\triangle ABC$ and $A'B'C'$ are equal. | 4. § 252. |
| 5. $\therefore \triangle ABC$ and $A'B'C'$ are either congruent or symmetrical. | 5. §§ 263, 264. |

268. Corollary. — If two sides of a spherical triangle are equal, the angles opposite these sides are equal.

In the spherical $\triangle ABC$ let $AB = AC$.

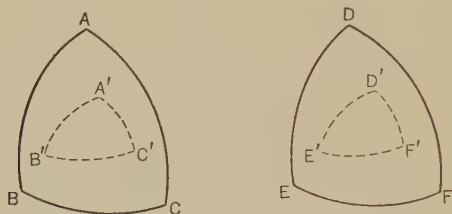
Draw great circle arc AD bisecting BC .

Then use § 267.



Proposition XI. Theorem

269. *Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetrical, if the three angles of one are equal respectively to the three angles of the other.*



Given: Spherical $\triangle ABC$ and DEF in which $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

To prove: $\triangle ABC$ and DEF are congruent or symmetrical.

Plan of Attack: class — congruent or symmetrical triangles.
method to be used — show that their polar triangles have their sides equal (§ 258);
 $\therefore$ their $\angle$ s are equal (§ 267). $\therefore$ sides of given $\triangle$ are equal (§ 258); etc.

Proof:

- | | |
|---|-----------------|
| 1. Draw $A'B'C'$ and $D'E'F'$ the polar $\triangle$ of ABC and DEF respectively. | 1. § 256. |
| 2. $\angle A$ is a supplement of $\widehat{B'C'}$ and $\angle D$ is a supplement of $\widehat{E'F'}$. | 2. § 258. |
| 3. $\therefore \widehat{B'C'} = \widehat{E'F'}$. | 3. § 315. |
| 4. Likewise, $\widehat{A'B'} = \widehat{D'E'}$ and $\widehat{A'C'} = \widehat{D'F'}$. | 4. Why? |
| 5. $\therefore \triangle A'B'C'$ and $D'E'F'$ are congruent or symmetrical. | 5. § 267. |
| 6. $\therefore \angle A' = \angle D'$, $\angle B' = \angle E'$, $\angle C' = \angle F'$. | 6. §§ 263; 264. |
| 7. $\therefore \widehat{BC} = \widehat{EF}$, $\widehat{AC} = \widehat{DF}$, and $\widehat{AB} = \widehat{DE}$. | 7. Why? |
| 8. $\therefore \triangle ABC$ and DEF are congruent or symmetrical. | 8. § 267. |

270. Theorem. — If two angles of a spherical triangle are equal, the sides opposite are equal.

Let $\triangle ABC$ be a spherical $\triangle$ in which $\angle B = \angle C$. Draw the polar $\triangle A'B'C'$. Then $A'B'$ and $A'C'$ are the respective supplements of $\angle C$ and $\angle B$ (§ 258) and are equal (§ 315). $\therefore \angle C' = \angle B'$ (§ 268). Then $AB = AC$ (§§ 258 and 315).

271. Theorem. — If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.

Let $\triangle ABC$ be a spherical $\triangle$ in which $\angle C > \angle B$. Draw the great circle arc DC meeting $\widehat{AB}$ in D making $\angle DCB = \angle B$. Then $\widehat{BD} = \widehat{DC}$ (§ 270). But $\widehat{CD} + \widehat{DA} > \widehat{AC}$ (§ 253). $\therefore \widehat{BD} + \widehat{DA} > \widehat{AC}$ (Ax. VIII). $\therefore \widehat{AB} > \widehat{AC}$ (Ax. VIa and VIII).



272. Theorem. — If two sides of a spherical triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.

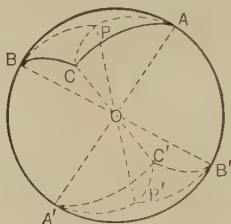
Use an indirect proof based upon §§ 270 and 271.

Exercises

1. Prove that if the four sides of a spheric quadrilateral are equal, its diagonals bisect each other.
2. Given the horizontal plane P and the vertical plane Q ; A is a point in the line of intersection of P and Q ; line l in Q and line m in P are each perpendicular to the line of intersection at A . Find the locus of points in space (a) 5'' from A and 3'' from P ; (b) equidistant from P and Q ; (c) equidistant from l and m ; (d) 5'' from A and equidistant from P and Q ; (e) 5'' from A and 3'' from Q .

Proposition XII. Theorem

273. Two symmetrical spherical triangles are equivalent.



Given: Spherical $\triangle ABC$ and the symmetrical spherical $\triangle A'B'C'$ formed by the planes of its sides passing through the center of the sphere.

To prove: $\triangle ABC$ equivalent to $\triangle A'B'C'$.

Plan of Attack: class — equivalent spherical triangles.

method to be used — show that $\triangle ABC$ and $A'B'C'$ are composed of the same number of parts which are congruent in pairs; then use axioms.

Proof:

- | | |
|--|-----------------|
| 1. Let P be the pole of the circle through A , B , and C . Draw the diameter POP' . Draw great circle arcs PA , PB , PC and $P'A'$, $P'B'$, and $P'C'$. | 1. §§ 307; 311. |
| 2. Then $\widehat{PA} = \widehat{PB} = \widehat{PC}$. | 2. § 231. |
| 3. But $\widehat{P'A'} = \widehat{PA}$, $\widehat{P'B'} = \widehat{PB}$, and $\widehat{P'C'} = \widehat{PC}$. | 3. §§ 319; 345. |
| 4. $\therefore \widehat{P'A'} = \widehat{P'B'} = \widehat{P'C'}$. | 4. Why? |
| 5. $\therefore \triangle PAB$ and $P'A'B'$ are isosceles and symmetrical. | 5. §§ 250; 263. |
| 6. $\therefore \triangle PAB = \triangle P'A'B'$. | 6. § 264a. |
| 7. Similarly $\triangle PBC = \triangle P'B'C'$ and $\triangle PAC = \triangle P'A'C'$. | 7. Why? |

$$8. \therefore \triangle PAB + \triangle PBC + \triangle PAC = \triangle P'A'B' + \triangle P'B'C' + \triangle P'A'C'.$$

$$9. \therefore \text{area } \triangle ABC = \text{area } \triangle A'B'C'.$$

10. But $\triangle A'B'C'$ is congruent to any other $\triangle$ which is symmetric with $\triangle ABC$.

11. $\therefore$ any two symmetrical spherical $\triangle$ are equal in area.

8. Why?

9. Why?

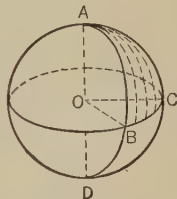
10. § 264*b*.

11. Why?

274. A **spherical degree** is the area of a birectangular spherical triangle having a vertex angle of one degree.

It is the unit used in measuring and comparing areas of spherical figures.

275. A **lune** is a spherical figure formed by two great semicircles which have the same endpoints; as lune $ABDC$.



Theorems Proved Informally

276. The area of a birectangular spherical triangle expressed in spherical degrees is equal to the number of degrees in the third angle of the triangle.

277. The area of a trirectangular spherical triangle is equal to 90 spherical degrees.

278. The area of a hemisphere is equal to 360 spherical degrees.

279. The area of a sphere is equal to 720 spherical degrees.

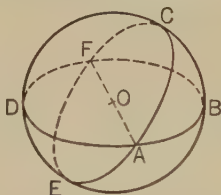
280. The area of a lune expressed in spherical degrees is equal to twice the number of degrees in the angle of the lune.

281. The area of a lune is to the area of the sphere as the angle of the lune is to four right angles.

282. The areas of two lunes are to each other as the number of degrees in their angles.

Proposition XIII. Theorem

283. *The area of a spherical triangle expressed in spherical degrees is numerically equal to its spherical excess.*



Given: Spherical $\triangle ABC$ whose spherical excess is denoted by E .

To prove: Area $\triangle ABC = E$ spherical degrees.

Plan of Attack: method to be used — compare area of triangle with areas of lunes whose angles are the angles of the triangle.

Proof:

- | | |
|--|--------------------|
| 1. Complete the great circles whose arcs are the sides of the spherical triangle. | 1. § 311. |
| 2. $\triangle ABC + \triangle BFC = \text{lune } ABFC =$
$2A$ spherical degrees. | 2. § 280. |
| 3. But $\triangle ADE = \triangle BFC$. | 3. §§ 263;
273. |
| 4. $\therefore \triangle ABC + \triangle ADE =$
$2A$ spherical degrees. | 4. Why? |
| $\triangle ABC + \triangle DAC = \text{lune } BCDA =$
$2B$ spherical degrees. | |
| $\triangle ABC + \triangle AEB = \text{lune } CAEB =$
$2C$ spherical degrees. | |
| 5. $\therefore 2\triangle ABC + (\triangle ABC + \triangle AEB +$
$\triangle ADE + \triangle DAC) = 2(A + B + C)$
spherical degrees. | 5. Why? |
| 6. But $\triangle ABC + \triangle AEB + \triangle ADE + \triangle DAC$
$= \text{hemisphere} = 360$ spherical degrees. | 6. § 278. |

- | | |
|---|----------|
| 7. $\therefore 2 \triangle ABC + 360$ spherical degrees = | 7. Why? |
| $2(A + B + C)$ spherical degrees. | |
| 8. $\therefore \triangle ABC + 180$ spherical degrees = | 8. Why? |
| $(A + B + C)$ spherical degrees. | |
| 9. $\therefore \triangle ABC = ((A + B + C) - 180)$ | 9. Why? |
| spherical degrees. | |
| 10. $\therefore \triangle ABC = E$ spherical degrees. | 10. Why? |

284. Corollary I. — The area of a spherical triangle is to the area of the sphere as the spherical excess of the triangle is to eight right angles. $\frac{\text{Area } \triangle}{\text{Area sphere}} = \frac{E}{720}$. Why?

285. Corollary II. — The area of a convex spherical polygon expressed in spherical degrees is numerically equal to its spherical excess.

286. Corollary III. — A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle.

287. Corollary IV. — A convex spherical polygon is equivalent to a lune whose angle is half the spherical excess of the polygon.

Exercises

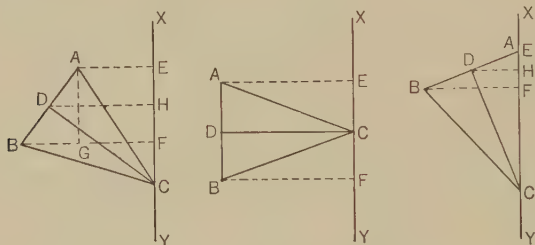
1. Find the ratio of the area of an equilateral spherical triangle, each of whose angles is 86° , to the area of a lune whose angle is 65° .

2. On the same sphere there is an equiangular spheric triangle, each of whose angles is 93° , and a lune whose angle is 70° . Find the ratio of the area of the triangle to the area of the lune.

3. Two angles of a spheric triangle are 110° and 50° . What must be the size of the third angle if the area of the triangle is one tenth the area of the entire sphere?

Proposition XIV. Theorem

288. *The area of the surface generated by the base of an isosceles triangle which revolves about an axis lying in its plane and passing through its vertex but not intersecting its surface, is equal to the projection of the base on the axis multiplied by the circumference of the circle the radius of which is the altitude of the triangle.*



Given: AB , the base, and CD , the altitude, of the isosceles $\triangle ABC$; line XY in the plane of $\triangle ABC$ passing through C but not intersecting $\triangle ABC$; EF the projection of AB on XY .

To prove: Area generated by $AB = EF \times 2\pi CD$.

Plan of Attack: method to be used — show that AB generates the lateral surface of a frustum of a cone (cone or cylinder) whose area may be represented by $EF \times 2\pi CD$.

Proof: I.

- | | |
|--|-----------------|
| 1. Draw $AG \perp BF$ and $DH \perp EF$. | 1. § 314. |
| 2. AB generates the lateral surface of a frustum of a cone of revolution. | 2. §§ 178; 196. |
| 3. $\therefore$ area $AB = AB \times 2\pi DH$. | 3. § 200. |
| 4. $\triangle ABG \sim \triangle CDH$. | 4. Why? |
| 5. $\therefore \frac{AB}{CD} = \frac{AG}{DH}$ or $\frac{AB}{CD} = \frac{EF}{DH}$. | 5. Why? |
| 6. $\therefore EF \times CD = AB \times DH$. | 6. § 360. |
| 7. $\therefore$ area $AB = EF \times 2\pi CD$ | 7. Why? |

II. If $AB \parallel XY$, AB generates the lateral surface of a right circular cylinder.

$$\begin{array}{ll} 8. \therefore \text{area } AB = AB \times 2\pi BF & 8. \text{ § 172.} \\ & = EF \times 2\pi CD. \end{array}$$

III. If side AC lies in XY , AB generates the lateral surface of a cone of revolution.

$$9. \therefore \text{area } AB = AB \times 2\pi DH. \quad 9. \text{ § 198.}$$

$$10. \text{ But } \triangle CDH \sim \triangle ABF. \quad 10. \text{ Why?}$$

$$11. \therefore \frac{AB}{CD} = \frac{AF}{DH} \text{ or } \frac{EF}{DH}. \quad 11. \text{ Why?}$$

$$12. \therefore EF \times CD = AB \times DH. \quad 12. \text{ Why?}$$

$$13. \therefore \text{area } AB = EF \times 2\pi CD. \quad 13. \text{ Why?}$$

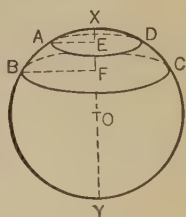
Exercises (*True or false?*)

1. Three planes tangent to the same sphere cannot be parallel.
2. Four spheres may be so placed that each is tangent to each of the other three.
3. A given point on a sphere is the pole of only one great circle.
4. At a given point on a sphere only one line can be drawn tangent to the sphere.
5. Through two points on a sphere only one great circle can be drawn.
6. All the tangents to a sphere at a point on the sphere lie in the same plane.
7. It is impossible to have a spherical triangle the sum of whose angles is 400° .
8. Two great circles of a sphere may intersect in only one point.
9. All the tangents to a sphere from a point outside the sphere form a conical surface.
10. If a point is at a quadrant's distance from the ends of a diameter of a sphere, it is the pole of every great circle through the points.

289. A **zone** is the portion of the surface of a sphere included between two parallel planes; as zone $ABCD$.

The *bases* of a zone are the circles of the sphere formed by the parallel planes; the *altitude* of the zone is the perpendicular distance between the planes, as EF .

A *zone of one base* is formed, if one of the parallel planes is tangent to the sphere; as zone XAD .



If the two parallel planes are both tangent to the sphere, the zone is the sphere.

290. An **arc generates a zone**. A zone may be generated by revolving an arc of a semicircle about its diameter as an axis.

If, in the above figure, the semicircle $XABY$ revolves about its diameter XY as an axis, $\widehat{AB}$ generates a zone with altitude EF ; $\widehat{AX}$ generates a zone of one base with altitude XE ; arc $XABY$ generates a sphere, or a zone with altitude XY .

291. Limits. In the accompanying figure the semicircle $XABCY$ has XOY as its diameter; D is the midpoint of $\widehat{AB}$, and DA and DB are chords; OR the distance of the chords from the center O .

If the figure $XABCY$ revolves about XY as an axis, $\widehat{AB}$ generates a zone and chords DA and DB generate curved surfaces whose combined area is less than the area of the zone. If now the number of chords inscribed is indefinitely increased by successive doubling, the surface generated by the chords increases in area and approaches as a limit the area of the zone generated by AB .



Also as the number of equal chords increases, the perpendicular OR increases in length and approaches the length of the radius as a limit.

In a similar manner, if half of a regular polygon of an even number of sides, as $XABCY$, is inscribed in a semicircle and the figure is revolved about the diameter XY , the area generated by the chords will approach as a limit the area of the sphere as the number of sides of the polygon is indefinitely increased; and the apothem OR will approach as a limit the radius of the sphere.

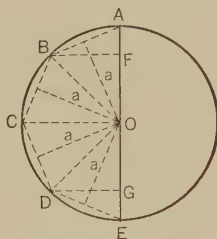
Interesting Relations

If a cone with its altitude equal to its diameter is filled with water and the water poured into a cylinder of the same dimensions, it is found that the cylinder may be filled by pouring the contents of the cone into the cylinder three times. If, however, a sphere of equal diameter is first placed within the cylinder, only one filling of the cone is required to fill the cylinder. This experiment leads to the following conclusions: the volume of the cone is one third of the volume of the cylinder; and the volume of the sphere is two thirds of the volume of the cylinder. Since the volume of the cylinder is $\pi r^2 \times 2r$, or $2\pi r^3$, we may infer that the volume of the sphere is $2/3$ of $2\pi r^3$, or $4/3\pi r^3$.

Archimedes (287–212 B.C.) a mathematician of Syracuse, on the island of Sicily, discovered many remarkable properties relating to the sphere and cylinder. He found that the volume of a sphere is two thirds of the volume of the circumscribed cylinder, and that the surface area of the sphere is also two thirds of the surface area of the circumscribed cylinder; that the surface area of the sphere is four times the area of one of its great circles, and is equal to the lateral area of the circumscribed cylinder.

Proposition XV. Theorem

292. *The area of a sphere is equal to the area of four great circles of the sphere.*



Given: The sphere generated by revolving the semicircle $ABCDE$ about the diameter AE as an axis; S denoting the surface of the sphere and R its radius.

To prove: $S = 4\pi R^2$.

Plan of Attack: method to be used — informal proof by inference; comparing the area of the sphere with the area generated by the revolution of half of a regular polygon.

Proof:

- | | |
|--|-----------|
| 1. Inscribe in the semicircle half of a regular polygon, $ABCDE$, of an even number of sides. Draw OB , OC , and OD . | 1. § 307. |
| 2. Draw a , the apothem to the sides, and AF , FO , OG , and GE , the projections of the sides on AE . | 2. § 314. |
| 3. Area generated by $AB = AF \times 2\pi a$.
area $BC = FO \times 2\pi a$.
area $CD = OG \times 2\pi a$.
area $DE = GE \times 2\pi a$. | 3. § 288. |
| 4. $\therefore$ area generated by $ABCDE =$
$(AF + FO + OG + GE) \times 2\pi a =$
$AE \times 2\pi a$. | 4. Why? |

5. If the number of sides of the polygon be indefinitely increased by successive doubling, the surface generated by the polygon will approach S and a will approach R as limits.	5. § 291.
6. But the surface generated by each successive polygon $= AE \times 2 \pi a$.	6. Proved.
7. It is reasonable, therefore, to conclude that $S = AE \times 2 \pi R = 2 R \times 2 \pi R$ or $S = 4 \pi R^2$.	7. By inference.

293. Corollary I. — The areas of two spheres are to each other as the squares of their radii, or as the squares of their diameters.

$$\frac{S}{S'} = \frac{4 \pi R^2}{4 \pi R'^2} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2} D)^2}{(\frac{1}{2} D')^2} = \frac{D^2}{D'^2}.$$

294. Corollary II. — The area of a spherical degree equals $\frac{4 \pi R^2}{720}$.

It follows from this corollary that the area of a spherical triangle, or of a spherical polygon $= E \times \frac{4 \pi R^2}{720}$ (§ 283) and the area of a lune $= 2 \angle A \times \frac{4 \pi R^2}{720}$ (§ 280).

295. Theorem. — The area of a zone equals the product of its altitude and the circumference of a great circle of the sphere; or $S = 2 \pi R h$.

This is proved in a manner similar to § 292. See § 291.

Exercises

1. The angles of a spheric triangle are 85° , 90° , and 140° ; the radius of the sphere is 6 inches. Find in square inches the area of the triangle.

2. Find the area of a zone included between parallel planes 4'' apart, the radius of the sphere being 6''.
3. Find the number of square centimeters in the surface of a globe which is one decimeter in diameter.
4. Find the cost, at \$2.50 a square foot, of gilding a hemispheric dome whose diameter is 40 feet.
5. Prove algebraically that the area of a zone of one base is equal to the area of a circle whose radius is the chord of the generating arc of the zone.
6. If the earth is a sphere of radius R , what is the area of the zone visible from a point whose height above the surface of the earth is H ?
7. What portion of the surface of the earth could be seen if one were lifted above the earth a distance equal to the radius?
8. Assuming that the earth is a sphere, show that one half the earth's surface is included between the parallels 30° N. and 30° S. latitude.
9. A cylinder whose altitude is equal to its diameter is inscribed in a sphere whose radius is r . Compare the total surface of the cylinder with that of the sphere.
10. Compute the number of square inches in the area of a triangle drawn on the surface of a sphere whose diameter is 6 inches, the angles of the triangle being $91^\circ 30'$, $110^\circ 30'$, and 135° .
11. Considering the earth as a sphere with a radius of 4000 miles, find the approximate area of the triangle on the earth's surface whose vertices are the north pole, a point in zero latitude and zero longitude and a point in zero latitude and 72° west longitude.
12. Prove that the area included between the concentric circles formed by a plane cutting two concentric spheres is constant for all positions of the plane.
13. On a sphere whose radius is 6 inches what is the area of a spheric triangle whose angles are 90° , 100° , and 110° ?

14. On the base of a right circular cone a hemisphere is constructed so that it lies outside the cone. The surface of the hemisphere is equal to the lateral surface of the cone. The radius of the hemisphere is r . Find the slant height of the cone.

15. Find the number of square inches in the area of a triangle drawn on a sphere whose diameter is 12 inches, the angles of the triangle being $85^{\circ} 11'$, $114^{\circ} 28'$, and $142^{\circ} 21'$.

16. A right circular cone, a right circular cylinder, and a sphere each have a radius r and an altitude $2r$. Find the ratio of the sum of the total surfaces of the cone and the cylinder to the surface of the sphere.

17. A lune whose angle is 40° is equivalent to a zone on the same sphere. Find the ratio of the altitude of the zone to the radius of the sphere.

18. The sides of a spheric triangle on a sphere of radius $12''$ are 67° , 53° , and 84° . Find the number of square inches in the area of the polar triangle.

19. The chord of the polar distance of a circle of a sphere is 18 inches. If the radius of the sphere is 16 inches, find the area of the zone cut from the sphere by the circle.

20. A right circular cylinder with hemispheric ends has its greatest length l and its diameter d . Find its surface.

21. A light is 18 inches from the center of a sphere whose radius is 6 inches. Find the area of the illuminated surface.

22. Find the number of square feet in the surface of a spheric triangle if its angles are 35° , 85° , and 110° and the radius of the sphere is 40 feet.

23. A right circular cone has its vertex in the surface of a sphere and its base is a section of the sphere made by a plane passing through the center. Find the ratio of the total surfaces of the two solids.

24. On a sphere whose radius is 5 feet the area of a zone is 20π square feet. Find the altitude of the zone.

25. If two angles of a spheric triangle, on a sphere whose radius is 10 feet, are 120° and 95° , find the value of the third angle if the area of the triangle is 80π square feet.

26. A spheric triangle on the earth's surface (regarded as a sphere) has one of its vertices at the north pole and the other two vertices on the equator. Find the size of each of its angles if the area of the triangle is equal to one half the area of a zone on the earth's surface whose altitude is equal to one half the radius.

27. An equilateral triangle with its inscribed circle is revolved about one of its altitudes as an axis. If one side of the triangle is 6, find the volume of the cone and the area of the surface of the sphere generated.

28. The area of a zone is 84π and its altitude is 7. Find the area of the spheric triangle on the same sphere if the sides of its polar triangle are 60° , 85° , and 95° .

29. A great circle of a sphere forms one base of a right cylinder, while the other base of the cylinder is tangent to the sphere. Find the ratio of the total surfaces of the two solids.

30. Find the ratio of the area of that portion of the earth's surface included between the meridians 45° W. longitude and 60° W. longitude to the portion included between the meridians 140° W. longitude and 145° E. longitude.

31. What fraction of the surface of a sphere is covered by a spheric triangle whose angles are 120° , 100° , and 40° ?

32. What is the area of a spheric triangle whose angles are 85° , 90° , and 95° on a sphere whose surface is 64 square inches?

33. Which has the larger area, a spheric triangle whose angles are 84° , 96° , and 50° , or a lune whose angle is 20° ?

34. Estimate the volume of a sphere by the prismatoid formula, § 155. In doing so, notice that $B = 0$, $b = 0$, $m = \pi R^2$, and $h = 2R$.

296. A **spherical pyramid** is the solid formed by a spherical polygon and the planes of its sides.

The *base* of the spherical pyramid is the spherical polygon, and its *vertex*, the center of the sphere. See figure for § 249.

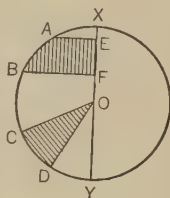
297. A **spherical wedge** is the solid formed by a lune and the planes of its arcs.

298. A **spherical segment** is the solid formed by a zone and the planes of its bases.

The *bases* of the segment are the sections of the sphere formed by the parallel planes; its *altitude* is the perpendicular distance between the parallel planes.

A spherical segment like a zone may have only one base.

If the semicircle revolves about XY as an axis, $ABFE$ will generate a spherical segment of two bases and AXE , a spherical segment of one base.



299. A **spherical sector** is the solid generated by revolving a sector of a circle about any diameter which does not cut the sector.

The *base* of the spherical sector is the zone generated by the arc of the sector.

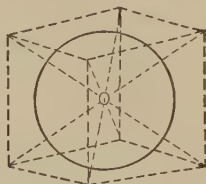
A spherical sector whose base is a zone of one base is sometimes called a *spherical cone*.

If the semicircle, § 298, revolves about XY as an axis, the circle sector COD will generate a spherical sector and DOY will generate a spherical cone.

300. Limits. — If a polyhedron is circumscribed about a sphere and the number of faces of the polyhedron is indefinitely increased, the surface of the polyhedron will approach the surface of the sphere as a limit and the volume of the polyhedron will approach the volume of the sphere as a limit.

Proposition XVI. Theorem

301. *The volume of a sphere is equal to one third the product of its radius and the area of its surface.*



Given: A sphere O , with its volume denoted by V , its radius by R , and its surface by S .

To prove: $V = \frac{1}{3} R \times S$, or $V = \frac{4}{3} \pi R^3$.

Plan of Attack: method to be used — informal proof by inference.

Proof:

- | | |
|--|---|
| <p>1. Consider a cube, or other polyhedron, circumscribed about the sphere. Denote its volume by V' and its surface by S'.</p> <p>2. Connect each vertex of the polyhedron with the center of the sphere, thus dividing the polyhedron into pyramids having the faces of the polyhedron as their bases and having the common vertex O and common altitude equal to R.</p> <p>3. Then the volume of each pyramid = $\frac{1}{3} R \times$ its base, and the sum of the volumes of all the pyramids = $\frac{1}{3} R \times$ the sum of all their bases; or $V' = \frac{1}{3} R \times S'$.</p> <p>4. By passing planes tangent to the sphere cutting off the vertices of the polyhedron another circumscribed polyhedron is formed whose surface and volume approach more nearly the surface and volume of the sphere.</p> | <p>1. § 241.</p> <p>2. § 307.</p> <p>3. Why?</p> <p>4. §§ 239, 241.</p> |
|--|---|

5. Continuing in this manner, the number of faces of the polyhedron may be indefinitely increased and its surface and volume may be made to approximate the surface and volume of the sphere. | 5. § 300.

6. For each successive polyhedron $V' = \frac{1}{3} R \times S'$. | 6. Proved.

7. Hence, we may conclude that $V = \frac{1}{3} R \times S = \frac{1}{3} R \times 4\pi R^2$ or $V = \frac{4}{3} \pi R^3$. | 7. By inference.

302. Corollary. — The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.

$$\frac{V}{V'} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi R'^3} = \frac{R^3}{R'^3} = \frac{(\frac{1}{2}D)^3}{(\frac{1}{2}D')^3} = \frac{D^3}{D'^3}.$$

303. Theorem. — The volume of a spherical pyramid is equal to one third the product of the radius of the sphere, by the area of the polygon that is its base. (See § 294.)

(Prove by inference in a manner similar to § 301.)

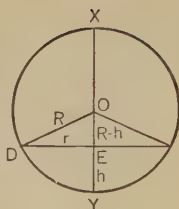
304. Theorem. — The volume of a spherical wedge is equal to one third the product of the radius of the sphere by the area of the lune that is its base. (See § 294.)

(The wedge may be divided into spherical pyramids.)

305. Theorem. — The volume of a spherical sector, or spherical cone, is equal to one third the product of the radius of the sphere by the area of the zone that is its base. $V = \frac{1}{3} R \times 2\pi rh = \frac{2}{3} \pi r^2 h$ (§ 295).

306. Volume of a spherical segment. A spherical segment of one base should be treated as the difference between the volume of a spherical sector (spherical cone) and the volume of the right circular cone.

Thus in the accompanying figure, if the semicircle XDY revolves about its diameter XY , the volume of the spherical segment generated by DEY equals the volume of the spherical sector, or spherical cone generated by ODY minus the volume of the right circular cone generated by ODE .



If $OY = R$, $EY = h$, and $DE = r$, then $DO = R$ and $OE = R - h$. Volume of segment $= \frac{2\pi R^2 h}{3} - \frac{\pi r^2 (R - h)}{3}$ (§§ 305 and 203). But $r^2 = R^2 - (R - h)^2 = 2Rh - h^2$.
 $\therefore V = \frac{2\pi R^2 h}{3} - \frac{\pi(2Rh - h^2)(R - h)}{3} = \frac{3\pi R h^2 - \pi h^3}{3} = \frac{\pi h^2}{3}(3R - h)$.

The use of this formula is not recommended.

Exercises

1. Find the entire surface of a hemisphere equivalent in volume to a sphere whose radius is 4 inches.
2. Find the surface of a sphere whose volume is $682\frac{2}{3}\pi$ cubic inches.
3. A ball a foot in diameter weighs 250 pounds. Find the diameter of a ball of the same material that weighs 128 pounds.
4. Find the volume of a spheric shell whose inner radius is 10 and whose outer radius is 20.
5. A solid glass ball 6 inches in diameter is expanded by a glass blower till the glass is an inch thick. Find the outer diameter of the hollow globe.
6. A sphere of lead 8 inches in diameter is melted and cast into a cone 8 inches high. Find the diameter of the base of the cone.

7. The volume of a sphere is $1774\frac{2}{3}\pi$ cubic inches. Find its surface.

8. A solid metal sphere whose radius is 6 in. is recast into a spheric shell; the cavity is spheric and has the same radius as that of the original sphere. Find the thickness of the shell.

9. Find a formula for the weight of a spheric shell, the inside radius being r , the thickness of the metal being t , and the weight of a cubic unit of the metal being w .

10. A right circular cylinder whose radius is 7 is equivalent in volume to a sphere whose surface is 616. Find the altitude of the cylinder. ($\pi = \frac{22}{7}$.)

11. A lead cylinder of revolution 10 centimeters long and 4 centimeters in diameter is melted and cast into a hemisphere. Find the radius of the hemisphere.

12. Assuming that the earth is a sphere with a radius of 4000 miles and that the crust is 30 miles thick, find the volume of the crust of the earth.

13. Find the volume of a hemisphere whose entire surface equals S .

14. Find the volume of a spheric wedge if the angle of the lune forming its base is 36° and the radius of the sphere is 14 in.

15. A spherical pyramid has for its base a spheric triangle whose angles are 90° , 80° , and 120° . If the radius of the sphere is 21 ft., what is the volume of the pyramid?

16. A spherical wedge is cut from a sphere whose surface is 400π sq. in. Find the volume of the wedge if the angle of lune forming its base is 45° .

17. Find the volume of a spheric sector whose base is a zone of altitude 3 in. on a sphere whose radius is 10 in.

18. A spherical segment of one base is cut from a sphere whose radius is 15 in. If the altitude of the segment is 3 in., what is the volume of the segment? (Read § 306.)

19. The center of each of two spheres whose common radius is 6 inches is at the surface of the other. Find the volume of the solid common to both spheres.

20. A hole 6'' in diameter is bored through a sphere 10'' in diameter, the greatest possible amount of material being removed. Find the volume of the part cut out.

21. A ball 10'' in diameter in rolling over a board drops into a hole 8'' in diameter. What fractional part of the ball is above the board?

22. Find the volume of the largest sphere which can be cut from a metal cone, the diameter of whose base is 12'' and whose slant height is 12''.

23. Find the surface and volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 3'' and whose altitude is 4''.

24. The angles of a spherical quadrilateral are 80° , 150° , 90° , and 130° . If the diameter of the sphere is 16, find the volume of the spherical pyramid whose base is the quadrilateral and whose vertex is the center of the sphere.

25. A square and an equilateral triangle are each circumscribed about a circle of radius r . The altitude of the triangle is parallel to a side of the square. The whole figure is revolved about this altitude as an axis. Find the volumes of the solids generated by (a) the square; (b) the triangle; and (c) the circle.

26. $ABCD$ is a square with AC a diagonal. BD is an arc having A as its center and AB as its radius. The figure is revolved about the side AD as an axis. Show that the volume of the solid generated by the sector DAB is equal to the volume of the solid generated by the triangle ABC .

27. A right circular cone is filled with water. A ball is placed so that it rests within the cone. The diameter of the cone is 3'' and its altitude is 2''; the diameter of the ball 2.4''. How much water will remain in the cone?

APPENDIX

FACTS FROM PLANE GEOMETRY

Assumptions

307. One straight line, and only one, can be drawn through two points.

308. Two points determine a straight line; hence, two straight lines intersect in only one point.

309. A straight line may be produced to any required length, or terminated at any point.

310. A straight line is the shortest line between two points.

311. A circle may be described with any point as a center and any given line as a radius.

312. Any figure may be moved from place to place without altering its size or shape.

313. Through any point outside a given line, one line, and only one, can be drawn parallel to a given line.

314. Through a given point in a plane, one perpendicular, and only one, can be drawn to a line in the plane.

315. Complements or supplements of equals are equal.

316. Through a given point in a given line a line may be drawn making with the given line an angle equal to a given angle.

Axioms

- 317.** I. If equals are added to equals, the sums are equal.
II. If equals are subtracted from equals, the remainders are equal.
III. Doubles of equals are equal; or, if equals are multiplied by equals, the products are equal.
IV. Halves of equals are equal; or, if equals are divided by equals, the quotients are equal.
V. Like powers, or like roots, of equals are equal.
VI. *a.* The whole is equal to the sum of its parts.
b. The whole is greater than any of its parts.
VII. Things equal to the same thing, or to equal things, are equal to each other.
VIII. A quantity may be substituted for its equal in any process.
IX. If, of three quantities, the first is greater than the second and the second is greater than the third, then the first is greater than the third.
X. If unequals are increased by, diminished by, multiplied by, or divided by positive equals, the results are unequal in the same order.
XI. The sum of the larger members of two or more inequalities is larger than the sum of the smaller members.
XII. If unequals are subtracted from equals, the results are unequal in *reverse* order.

Congruent or Equal Figures

- 318.** All straight angles are equal; and all right angles are equal.
319. Vertical angles are equal.
320. Two triangles are congruent: (*a*) if two sides and the included angle of one respectively equal two sides and the

included angle of the other; (b) if a side and the two adjoining angles of one respectively equal a side and the two adjoining angles of the other; (c) if the three sides of one are respectively equal to the three sides of the other.

321. Two right triangles are congruent (a) if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other; (b) if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other; (c) if a leg and an acute angle of one are equal to a leg and corresponding acute angle of the other.

322. Corresponding parts of congruent figures are equal.

Perpendicular Bisectors

323. Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

324. The perpendicular bisector of a line segment is the locus of points equidistant from the ends of the line.

325. If two circles intersect, the line of centers is the perpendicular bisector of the common chord.

326. A radius to the midpoint of an arc is the perpendicular bisector of the chord which subtends the arc.

327. A circle may be circumscribed about any triangle (or drawn through three points not lying in a straight line); or about any regular polygon.

Right Triangles

328. In any right triangle, if the altitude to the hypotenuse is drawn, (a) the altitude is the mean proportional between the segments of the hypotenuse, (b) each leg is the mean

proportional between the whole hypotenuse and the adjacent segment of the hypotenuse, (c) the product of the two legs equals the product of the hypotenuse and the altitude on the hypotenuse.

329. The sum of the squares of the two legs of a right triangle equals the square of the hypotenuse.

330. In a 30° - 60° right triangle the hypotenuse is double the side opposite the 30° angle.

331. An angle inscribed in a semicircle forms a right triangle with the diameter which determines the semicircle.

Parallels

332. Two lines are parallel, if cut by a transversal so that (a) the alternate interior angles are equal (converse); (b) the corresponding angles are equal (converse); (c) the two interior angles on the same side of the transversal are supplementary (converse).

333. In a parallelogram, (a) the opposite sides are equal, (b) the opposite angles are equal, (c) the diagonals bisect each other, and (d) a diagonal divides the parallelogram into two congruent triangles.

334. A line parallel to one side of a triangle cuts the other two sides proportionally (converse).

335. A line connecting the midpoints of two sides of a triangle is parallel to the third side and equals one half the third side.

336. Two straight lines in the same plane perpendicular to the same line are parallel.

337. In a plane lines perpendicular to intersecting lines meet.

338. A straight line perpendicular to one of two parallel lines is perpendicular to the other.

339. A quadrilateral is a parallelogram, (a) if the opposite sides are parallel; (b) if the opposite sides are equal; (c) if two opposite sides are equal and parallel.

340. The line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the two bases and is equal to one half the sum of the bases.

Bisectors of Angles

341. The bisector of an angle is the locus of points equidistant from the sides of the angle.

342. The bisector of the vertex angle of an isosceles triangle is a median and an altitude.

343. A circle can be inscribed in any triangle.

344. The radii of a regular polygon bisect the angles of the polygon, and an apothem bisects the central angle.

Circles

345. Equal chords (or equal central angles) in a circle or in equal circles, have equal arcs (converse).

346. Equal chords are equally distant from the center (converse).

347. If two chords intersect within a circle, the product of the segments of one equals the product of the segments of the other.

348. A central angle is measured by its intercepted arc.

349. A diameter perpendicular to a chord bisects the chord and the arcs which the chord subtends.

350. A circle may be circumscribed about any regular polygon, and a circle may also be inscribed in it.

351. If at the midpoints of the arcs joining the adjacent points of contact of the sides of a regular circumscribed polygon tangents are drawn, a regular polygon of double the number of sides will be formed.

Tangents and Secants

352. A tangent to a circle at a given point is perpendicular to the radius at that point.

353. Two tangents to a circle from an external point are equal and the line joining the point to the center bisects the angle formed by the tangents.

354. If a tangent and a secant are drawn to a circle from a point, the tangent is the mean proportional between the whole secant and its external segment.

Angles and Sum of Angles

355. The sum of the angles of a triangle is equal to two right angles.

356. The sum of the angles of a polygon of n sides equals $(n - 2)2 \text{ rt. } \angle$.

357. An interior angle of a regular polygon of n sides equals $\frac{(n - 2)2 \text{ rt. } \angle}{n}$.

358. A central angle of a regular polygon of n sides equals $\frac{4 \text{ rt. } \angle}{n}$.

Proportion and Similar Triangles

359. If the second terms of two equal ratios are equal, the first terms are also equal (converse).

360. In a proportion the product of the extremes equals the product of the means.

361. Two triangles are similar, if they (*a*) are mutually equiangular; (*b*) have an angle of one equal to an angle of the other and the including sides proportional; (*c*) have their corresponding sides proportional; (*d*) are right triangles and have an acute angle of one equal to an acute angle of the other.

362. A line parallel to the base of a triangle cuts off at the vertex a triangle similar to the given triangle.

363. Perimeters of similar polygons are proportional to any two corresponding lines of the polygons. (In a series of equal ratios the sum of the first terms divided by the sum of the second terms equals any first term divided by its second term.)

364. Circumferences of two circles are proportional to their radii, or to their diameters.

365. Corresponding altitudes of two similar triangles are proportional to any two corresponding sides.

Areas

366. The area of a rectangle, or of a parallelogram, equals the product of its base and its altitude.

367. The area of a triangle equals one half the product of its base and its altitude.

368. The area of a trapezoid equals one half the product of its altitude and the sum of its bases; or the product of the altitude and the median of the trapezoid.

369. The areas of two similar polygons are to each other as the squares of any two corresponding lines.

370. The area of a regular polygon equals one half the product of its perimeter and its apothem.

371. The areas of two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

372. The area of a circle equals one half the product of its circumference and its radius; or the square of its radius multiplied by π .

373. The area of a regular hexagon equals the sum of the areas of the six equilateral triangles formed by drawing the radii of the polygon.

374. The area of an equilateral triangle equals one fourth the square of one side multiplied by $\sqrt{3}$.

Unequals

375. The perpendicular is the shortest line that can be drawn from a point to a straight line.

376. If two triangles have two sides of one respectively equal to two sides of the other, and the third sides unequal, the triangle which has the greater third side has the greater included angle.

377. If two oblique straight lines drawn from a point to a straight line meet the line at unequal distances from the foot of the perpendicular drawn from the point to the line, the more remote is the greater.

378. If in the same circle, or in equal circles, two arcs are unequal, their chords are unequal, the greater arc having the greater chord (converse).

379. If in the same circle, or in equal circles, two chords are unequal, the greater chord is nearer the center (converse).

Special Figures

380. In an *isosceles trapezoid*, (a) the base angles are equal; (b) the diagonals are equal; (c) the median equals one half the sum of the bases.

381. In a *rectangle*, (a) the angles are right angles; (b) the diagonals are equal; (c) all the properties of a parallelogram are present; (d) a diagonal is the diameter of the circumscribed circle.

382. In a *rhombus* or a *square*, (a) all the sides are equal; (b) the diagonals are perpendicular; (c) the diagonals bisect the angles; (d) the properties of a parallelogram are present; (e) area equals one half the product of its diagonals.

383. In an *equilateral triangle*, (a) all the angles are equal; (b) the altitude equals one half the side multiplied by $\sqrt{3}$; (c) the apothem equals half the radius; (d) the apothem equals one third the altitude; (e) the bisector of an angle is also a median and an altitude.

384. In a *regular hexagon*, (a) the radii divide the polygon into six equilateral triangles; (b) the radius equals a side of the polygon; (c) the apothem equals half the side multiplied by $\sqrt{3}$.

Definitions

385. A *right angle* is one of the angles formed by a line that meets another line so as to make two equal adjacent angles.

386. A *perpendicular* to a line is a line that makes a right angle with the given line.

387. A *bisector* of a geometric figure divides it into two equal parts.

388. *Parallel lines* are lines in the same plane that cannot meet, however far extended.

389. *Congruent figures* are figures that can be made to coincide.

390. A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.

391. A *rectangle* is a parallelogram having four right angles.

392. A *rhombus* is a parallelogram having four oblique angles and four equal sides.

393. A *square* is a rectangle having four equal sides.

394. A *trapezoid* is a quadrilateral with only one pair of sides parallel.

395. A *circle* is a closed plane curve, all points of which are equidistant from a fixed point of the plane.

396. A *locus* is a geometric figure containing all the points, and only those points, which fulfill a given requirement.

397. *Congruent polygons* are mutually equiangular and mutually equilateral.

398. *Similar polygons* are polygons having their corresponding angles equal, and their corresponding sides proportional.

399. A *regular polygon* is both equilateral and equiangular.

400. The *center* of a regular polygon is the common center of the circumscribed and inscribed circles.

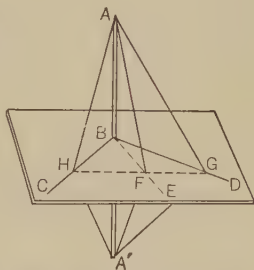
SUGGESTIONS AND HELPS

BOOK I

In the following pages will be found reasons omitted in the text-proved propositions; also, suggestions for the solutions of the more difficult exercises of the text.

Page 4

Proposition III. In the study of this theorem it is suggested that the pupil make a model to represent the figure. On a small piece of cardboard draw lines BC , BD , BE , and HFG . Through a small pinhole at B insert a match to represent ABA' . Make pinholes at H , F , and G ; notch the match at A and A' ; pass a thread from notch A through H to notch A' ; also pass threads from A through F to A' and from A through G to A' . Study this model while reading the proof in the text.



3. Construction and given.
6. Proved; identity; proved.
7. § 320a.
9. Construction.

Page 7

Proposition IV. 3. § 10.

6. § 20.
8. § 10.

Page 8

Proposition V. 2. § 10.

- | | |
|------------|-------------------|
| 6. § 386. | 10. Construction. |
| 8. § 320a. | 12. Construction. |
| 9. § 322. | 14. § 4. |

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Exercise. Given planes AF , CB , and ED intersecting in lines AB , CD , and EF .

FIG. I. Suppose AB and EF meet in G . Then point G is in plane CB and also in plane ED (§ 11b). $\therefore G$ lies in CD (§ 9b), etc.

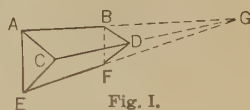


Fig. I.

FIG. II. $AB \parallel CD$. Then $AB \parallel EF$, since we proved in Figure I that, if AB and EF meet in a point, CD also passes through the point. Likewise $CD \parallel EF$.

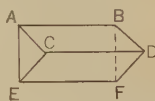


Fig. II.

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Proposition VI (I). 1. Given.

4. § 320a.

5. § 322.

(II). 1. § 4.

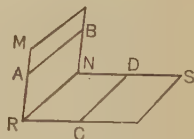
3. $BE > BC$, given; substitute BF for its equal BC .

6. Substituting AC for its equal AF in statement 4.

Page 11

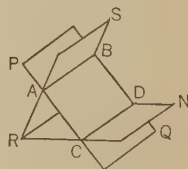
Exercise 3. Given $AB \parallel CD$.

Use §§ 14 and 16.

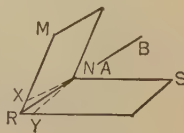


Exercise 4. Given plane $PQ \parallel$ the intersection of planes RN and RS .

Use §§ 14 and 16.



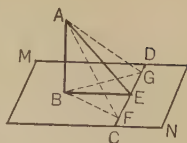
Exercise 5. Through AB and the point N in the line RN pass a plane, intersecting MN in NX and RS in NY (§§ 4, 10). Then $AB \parallel NX$ and $AB \parallel NY$ (§ 16). Use § 313.



Exercise 6. Given: $AB \perp$ plane MN .
 $BE \perp CD$ in MN .

Prove: $AE \perp CD$.

On CD lay off $EF = EG$; draw AF , AG , BF , and BG . Show that points A and E are equidistant from F and G , hence § 323.



Page 12

Proposition VII. 3. Given.

5. Construction.

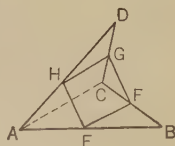
10. § 9c.

Page 13

Exercise 1. Given: quadrilateral $ABCD$, in which AB and CB are in one plane and AD and CD in another plane. E , F , G , and H are midpoints of AB , BC , CD , and AD .

Prove: $EFGH$ is a $\square$.

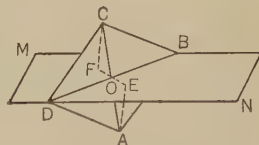
Use § 335.



Exercise 3. Given: $\square ABCD$ with diagonals AC and BD . Plane MN passing through BD . $AE \perp MN$ and $CF \perp MN$.

Prove: $AE = CF$.

Hint: Show that AE , CF , AC , and FE are in a plane, then use plane geometry.



Exercise 4. Use § 31.

Page 15

Proposition IX. 3. Given.

6. § 20.

Page 16

Proposition X. 4. § 9c.

6. § 20.

Page 18

Proposition XI. 2. Construction; given.

4. §§ 333a; 390.
9. § 333a.
10. § 320c.
11. § 322.

Page 19

Exercise 4. See figure for Exercise 6, page 11.

Page 23

Proposition XIII. 5. § 386.

Page 24

Proposition XIV. 3. Given.

6. § 386.
8. § 20.

Page 25

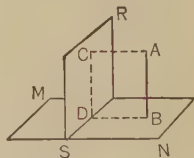
Exercise 2. Use §§ 14 and 57.

Exercise 5. Given: $AB \perp$ plane MN

$AB \parallel$ plane RS .

Prove: $RS \perp MN$.

Hint: Through AB pass a plane intersecting RS in CD .

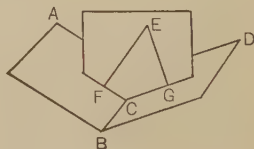


Page 26

Proposition XV. 4. Since it coincides with AB which is $\perp RS$.

Page 28

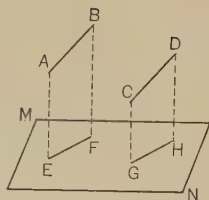
Exercise 6. Use §§ 57 and 62.



Exercise 7. Given: AB and CD oblique to plane MN . $AB \parallel CD$. EF and GH the projections of AB and CD on MN .

Prove: $EF \parallel GH$.

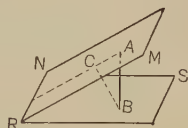
Hint: Show that $\angle BAE$ and $\angle DCG$ have their sides parallel; etc.



Page 29

Exercise 13. Given: intersecting planes MN and RS ; $AB \perp RS$; AC the projection of AB on MN .

Prove: $AC \perp RN$.



Exercise 14. Use the figure for Exercise 6, page 28.

Given: dihedral $\angle A-BC-D$, point E within the $\angle$. $EF \perp$ plane AB and $EG \perp$ plane BD .

Prove: $\angle FEG$ is the supplement of the plane $\angle$ of $\angle A-BC-D$.

Hint: Show that $\angle FCG$ is the plane $\angle$ of $\angle A-BC-D$ and that $\angle FEG + \angle FCG = 2 \text{ rt } \angle$.

Exercise 15. Given: dihedral $\angle A-BC-D$; point F in plane BD ; $EF \perp BD$ and $FG \perp AB$.

Prove: $\angle EFG =$ a plane $\angle$ of $\angle A-BC-D$.

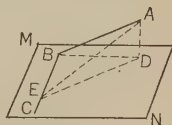
Hint: Through EF and FG pass a plane intersecting BD and AB in lines FH and HG . Show that $\angle FHG$ is a plane $\angle$ of $\angle A-BC-D$ and that $\angle EFG = \angle FHG$.



Exercise 20. Given: the right $\angle ABC$ with side BC in plane MN . $\angle DBC$ the projection of $\angle ABC$ on MN .

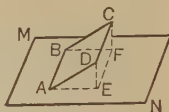
Prove: $\angle DBC$ is a rt $\angle$.

Hint: On BC lay off $BE = AD$, draw AE and ED . Show that $\triangle EBD = \text{rt. } \triangle ADB$. In $\triangle ADE$, $ED^2 = ?$ In $\triangle ABE$, $AB^2 = ?$



Exercise 21. Let $AEFB$ be the projection of square $ABCD$ on plane MN .

Show that $AEFB$ is a $\square$ by proving $AE = BF$ and $AE \parallel BF$. Then show that its $\angle$ s are rt. $\angle$ s, etc. What plane $\angle$ measures the dihedral $\angle$ formed by the planes?



Page 30

Proposition XVI. 2. $AC = BC$, given; $PC = PC$; $\angle ACP = \angle PCB$ (§ 9c). § 320a.

3. § 322.

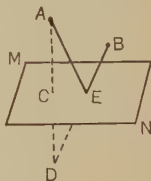
5. § 320c. $AP = PB$, given; $PC = PC$; $AC = CB$, given.

7. § 386.

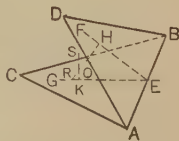
9. Every point in MN is equidistant from A and B and every point equidistant from A and B lies in MN .

Page 31

Exercise 4. Show that E is the required point by taking any other point in MN such as F . Show that $BE + ED < BF + FD$. Why? Then $BE + AE < BF + AF$. Why?

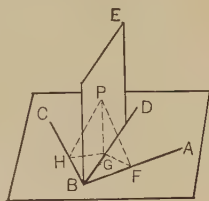


Exercise 11. Let A, B, C , and D be four points not all in the same plane. Pass planes through A, B, C and A, B, D . At E , the midpoint of AB , pass plane $\perp AB$. Show that this plane FEG contains H and K , the centers of circles circumscribed about $\triangle ABD$ and $\triangle ABC$. In plane FEG draw $HR \perp HE$ and $KS \perp KE$. These will intersect in some point O which is the required point. Plane $FEG \perp$ planes ABC and ABD (57), $\therefore HR \perp$ plane ABD and $KS \perp$ plane ABC (§ 59). Use § 31.



Exercise 12. Given BD bisects $\angle ABC$; plane EB through BD is $\perp$ plane ABC .

(a) Through P , any point in EB , pass plane $PGF \perp BA$, intersecting EB in PG . Then $PG \perp$ plane ABC (§§ 57, 62). Likewise a plane through $P \perp BC$ will intersect EB in a line $\perp ABC$, or in line PG (§ 26). $AB \perp PF$ and GF , also $BC \perp PH$ and HG (§ 9c). Then $PF = PH$. Why?



(b) Let point P be equidistant from AB and BC . Through P pass planes $\perp AB$ and BC . Let these planes intersect in PG . Show that $PG \perp$ plane ABC and that G is in the bisector BD , hence PG lies in plane EB (§§ 57, 62, 60, etc.).

Page 32

Proposition XVII. 6. Given MQ bisects $\angle N-MC-R$ (§ 53).

7. § 321*b*. $PC = PC$ and $\angle ACP = \angle BCP$.

8. § 322.

11. See statements 3–5.

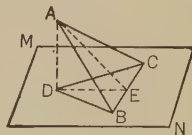
12. § 321*a*. $PA = PB$, given; and $PC = PC$.

15. § 387. There is only one bisector of an $\angle$.

16. All points in MQ are equidistant from the faces and all points equidistant from the faces are in MQ .

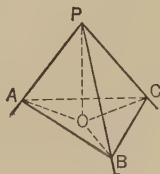
Page 33

Exercise 4. Let DCB be the projection of the $\triangle ABC$ on MN . Show that $\angle AED$ is a plane $\angle$ of the dihedral $\angle A-BC-M$.

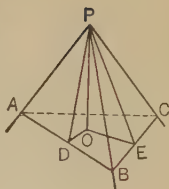


Page 34

Exercise 1. Let planes POA and POB , bisecting the dihedral angles whose edges are PA and PB , intersect in PO . Using § 71, show that PO must also be in the plane bisecting the dihedral $\angle$ whose edge is PC .



Exercise 2. Let planes POD and POE , passing through the bisectors of the face angles APB and BPC , be $\perp$ planes APB and BPC and intersect in line PO . By Exercise 12, page 31, show that PO must lie in the plane $\perp$ to face PAC and containing bisector of face $\angle APC$.



Page 35

Proposition XVIII. 3. § 320a. $FP = FP$; $PH = PE$, construction; $\angle APB = \angle APD$, construction (§ 322).

Page 36

Proposition XIX. 3. Since each side of polygon $ABCDE$ is the base of a $P\Delta$ and also of an $O\Delta$, the number of $P\Delta$ = the number of $O\Delta$ (§ 355).

Page 37

Proposition XIX. 7. The sum of the angles about a point in a plane is 4 rt. $\angle$.

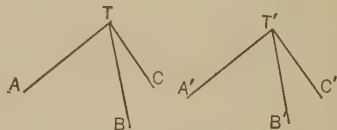
8. Substituting 4 rt. $\angle$ for its equal in statement 6.

Exercise 5. From the conclusion proved in § 75, by the use of axioms, show that $\angle APB > \angle APC - \angle BPC$.

Exercise 6. Given: $\angle ATB = \angle A'T'B'$, $\angle ATC = \angle A'T'C'$, dihedral $\angle TA =$ dihedral $\angle T'A'$.

Prove: $T-ABC = T'-A'B'C'$.

Prove by superposition. Place $\angle A'T'C'$ on its equal $\angle ATC$. The face $A'T'B'$ will fall along face ATB . Why? Etc.



Exercise 7. Given: $PA = PB = PC$, also $OA = OB = OC$. X is any point in PO .

Prove: X is equidistant from PA , PB , and PC .

Draw XD , XE , and $XF \perp PA$, PB , and PC . Show that rt. ΔPDX , PEx , and PFX are congruent.

Page 38

Proposition XX. 2. § 320a. $TD = T'D'$, $TE = T'E'$, construction; $\angle ATB = \angle A'T'B'$, given; etc.

4. § 320c.

6. § 321c. $DG = D'G'$, construction; and $\angle GDH = \angle G'D'H'$ (§ 322).

7. § 320a. $DH = D'H'$, $DK = D'K'$, and $\angle HDK = \angle H'D'K'$ (§ 322).

8. § 320c. $HG = H'G'$, $GK = G'K'$, $HK = H'K'$ (§ 322).

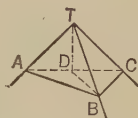
9. § 322.

Page 40

Exercise 2. Given $\angle ATB = \angle BTC$.

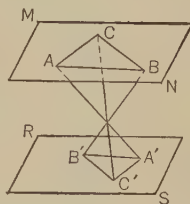
Prove dihedral $\angle TA = \angle TC$.

Hint: Let TD bisect $\angle ATC$. Compare trihedral $\angle T-ABD$ and trihedral $\angle T-DBC$, using § 77.



Exercise 4. Given planes $MN \parallel RS$ with point P between the planes. The lines AA' , BB' , and CC' through P intersect MN and RS in points A, B, C , and A', B', C' .

Prove $\triangle ABC \sim \triangle A'B'C'$. Use §§ 42; 361a.

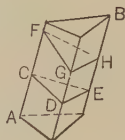


Page 44

Exercise 2. Given: prism AB with right section CDE an equilateral triangle. FGH a section with $GH \parallel DE$.

Prove: $\triangle FGH$ is isosceles.

Hint: Through GH pass another right section XGH , etc.



BOOK II

Page 45

Proposition I. 1. § 12.

4. § 333a.

Page 47

Proposition II. 8. Statements 5-7; § 389.

9. § 308.

Page 48

Proposition III. 1. § 89; identity.

4. § 332b. $EA \parallel E'A'$ and $LF \parallel L'F'$ (§§ 83 and 390).

8. Axiom I.

9. Axiom VIa.

Page 49

Proposition IV. 6. The perimeter of a figure is the sum of its sides.

7. Substituting p for its equal in 5.

Page 50

Exercise 1. In a rhombus the sides are equal and parallel. The diagonals are $\perp$ and bisect each other. The area equals half the product of its diagonals.

Exercise 2. The area of an equilateral $\triangle$ equals one fourth the square of a side multiplied by $\sqrt{3}$.

Exercise 3. The area of a regular hexagon equals the sum of the areas of the equilateral $\triangle$ formed by drawing the radii of the polygon.

Page 51

Exercise 5. Using the figure for § 103, let diagonals AG and FD intersect in O so that $FO = OD$ and $GO = OA$. Show that $AD = FG$ and $AD \parallel FG$, etc.

Page 59

Proposition VII. 3. Axiom VIa.

5. Axiom VIII. Substituting 2 $ABC-D$ for BE in statement 4.

Page 60

Proposition VIII. 3. § 87.

5. Axiom I.

6. Substituting the whole for the sum of its parts in **5**.

Page 61

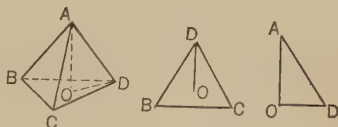
Exercise 4. The volume depends upon the area of the end. Use § 369.

Exercise 5. (a) Show that face $ABFE \perp$ face $BFGC$ and that $EF \perp BF$. $\therefore EF \perp BFGC$, etc.

(d) Consider $BFGC$ as the base and AB as a lateral edge.

Page 63

Exercise 1. In the equilateral $\triangle BCD$, OD is a radius of the circumscribed $\odot$. In an equilateral $\triangle$ the radius = two thirds of the altitude of the $\triangle$.

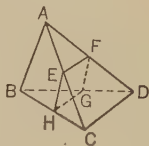


The altitude of the $\triangle$ = one half the side multiplied by $\sqrt{3}$.

2. Given: plane $EFGH \parallel$ edges AB and CD .

Prove: $EFGH$ is $\square$.

3. Use § 335.



Page 64

Proposition IX. 3. § 12.

7. Axiom VII.

Page 66

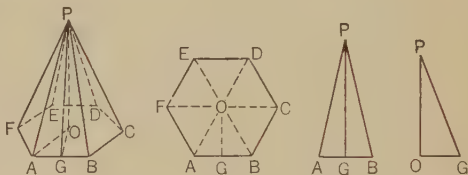
Proposition X. 3. Axiom I.

4. Substituting the whole for the sum of its parts.

Page 67

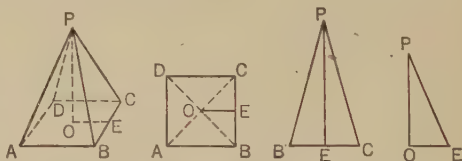
Exercise 2. The section formed by the plane determined by the altitude and slant height is an isosceles trapezoid with upper base 4 and lower base 8. Drop $\perp$ from the ends of the upper base and solve the right triangles formed.

Exercise 4. OG is the altitude of one of the six equilateral triangles formed by drawing the radii of the base hexagon. $\triangle ABP$ represents a



lateral face, with PG a slant height. PO is the altitude of the pyramid. Hence to find the slant height use $\triangle POG$.

Exercise 5. Figure $ABCD$ represents the square base. $\triangle PBC$ represents a lateral face, with slant height PE . PO is the altitude of the pyramid and $\triangle POE$ is the triangle formed by the altitude, slant height, and apothem of the base. Use equation: total surface = area of base + lateral area; and find slant height PE . Find OE . Then in the right $\triangle POE$ find PO or h .

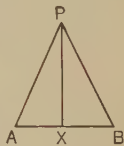


Exercise 7. In each case let $\triangle PAB$ represent a lateral face, PA a lateral edge, AB the edge of the base, and PX the slant height.

The area of an equilateral $\triangle$ = one fourth the square of a side multiplied by $\sqrt{3}$.

The area of a square = the square of one of its sides.

The area of a regular hexagon = the sum of the areas of the six equilateral triangles formed by drawing the radii of the hexagon.



Page 69

Proposition XI. 8. § 94.

9. Evident from statements 3 and 8.

11. Axiom VII.

14. Axioms IV and VII.

Page 69

Figures for regular pyramids. In all figures PO is the altitude of the pyramid, PX is the slant height, PA the lateral edge, O is the center of the regular polygon forming the base, OA is a radius of the base, and OX is an apothem of the base.

If the base is an equilateral $\triangle$, $OA = \frac{2}{3}$ of an altitude of the triangle ABC and $OX = \frac{1}{3}$ of an altitude. The altitude of an equilateral $\triangle = \frac{1}{2}$ the side multiplied by $\sqrt{3}$.

If the base is a square, $OA = \frac{1}{2}$ the diagonal of the square, $OX = \frac{1}{2}$ of a side of the square.

If the base is a regular hexagon, $OA =$ one side of the equilateral $\triangle OAB$, $OX =$ the altitude of the equilateral $\triangle OAB$.

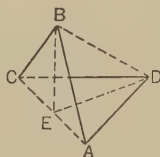
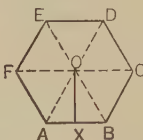
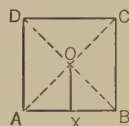
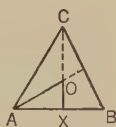
$\triangle PAB$ represents one lateral face of any regular pyramid, PA a lateral edge, PX a slant height, AB an edge of the base.

$\triangle POA$ represents the right $\triangle$ formed by the lateral edge PA , the altitude PO , and the radius of the base OA .

$\triangle POX$ represents the right $\triangle$ formed by the altitude PO , the slant height PX , and the apothem of the base OX .

Exercise 3. (a) AB , BC , CD , and DA are sides of a square; plane $ABC \perp$ plane ACD . If each edge of the square $ABCD$ equals e , show that BD also equals e .

(b) Show that BE is the altitude and ACD the base of the tetrahedron.



Page 70

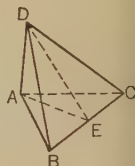
Proposition XII. 3. Axiom I.

4. Substituting the whole for the sum of its parts.

Page 71

Exercise 3. (a) Show that $\angle DEA$ is the plane angle and find the size of $\angle DEA$.

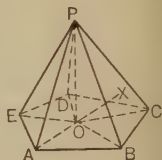
Exercise 4. (c) Consider EBC as the base of a pyramid and A as its vertex; find the altitude of the pyramid.



Page 72

Exercise 5. Draw PO , OB , OC , etc. Then the volume of $P-ABCDE = \text{volume } O-PBC + O-PAB + O-PCD + \text{etc.}$

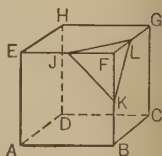
Exercise 8. Draw only one pyramid and from its surface and volume determine what the surface and volume of the entire solid is.



Exercise 10. See § 340.

Page 74

Exercise 25. How many pyramids equal to $F-JKL$ will be removed? Find the area of $\triangle JKL$; then the total surface of the solid = ? Find the volume of the pyramid $F-JKL$. When all these pyramids are removed, what is the volume of the remaining solid?



Exercise 26. Show that $\triangle BEG = \triangle BED = \triangle BDG = \triangle EDG$. Find the area of $\triangle BEG$. The total area will then equal what? Find the volume of pyramid $E-ABD$. How many pyramids are removed by the planes? What is the volume of the remaining solid?

BOOK III

Page 83

Proposition I. 1. § 12.

5. Since DE coincides with a line that is $\parallel CF$.

Page 84

Proposition II. 4. § 339c.

5. § 333a.

8. Corresponding parts of congruent figures can be made to coincide.

Page 86

Proposition III. 2. § 102.

5. § 102.

Page 88

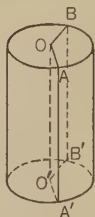
Proposition IV. 2. § 130.

5. § 130.

Page 90

Exercise 7. If the arc $AB = 144^\circ$, the arc $AB =$ what part of the circumference of the circular base?

If the arc $AB = 144^\circ$, the sector $AOB =$ what part of the area of the circular base?



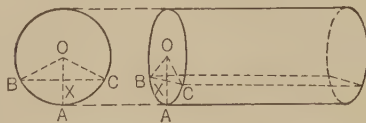
Exercise 12. How many inches in the length of OA ? OB ?

How many inches in the length of XA ? OX ?

How large is $\angle OBX$?

$\angle BOX$? $\angle BOC$? Sector $OBAC =$ what part of the area of the circle? What is the area of the sector $OBAC$?

What is the area of the $\triangle OBC$? What is the area of the segment BAC ? What is the volume of the gasoline in the tank?



Page 92

Exercise 2. Subtract the volume of the cylindric hole from the volume of the hexagonal prism.

Page 94

Proposition VI. 3. § 12.

5. § 398.

7. Radii of equal circles are equal.

Page 96

Proposition VII. 2. § 143.

5. § 143.

Page 98

Figures for various exercises.

When the base of the circumscribed pyramid is an equilateral $\triangle$. Altitude of $\triangle = \frac{1}{2}$ of a side multiplied by $\sqrt{3}$. Area of $\triangle = \frac{1}{4}$ the square of a side multiplied by $\sqrt{3}$. $OA = \frac{2}{3}$ of the altitude of the $\triangle$; $OX = \frac{1}{3}$ of the altitude.

When the base of the pyramid is a square. $OA = \frac{1}{2}$ the diagonal; $OX = \frac{1}{2}$ the side of the square.

When the base of the pyramid is a regular hexagon. The area of the hexagon = the sum of the areas of the six equilateral triangles formed by drawing the radii of the polygon. $OA =$ a side of the hexagon; $OX =$ an altitude of one of the equilateral triangles.



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Exercise 10. Use formula $h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$,

where $a, b, c =$ the three sides and $s = \frac{1}{2}(a + b + c)$.

Page 100

Proposition VIII. 2. § 146.

5. § 146.

Page 102

Exercise 11. Show that $r = \frac{3}{5} R$. Use the equation :

Volume of the large cone minus the volume of the small cone = 294.



BOOK IV

Page 108

Proposition I. 4. § 386.

Page 111

Proposition II. 2. Radii of a circle are equal.

5. Reasons 2-4.

Page 112

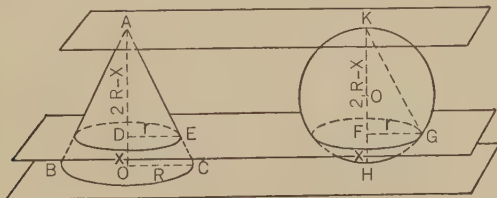
Proposition III. 2. Given.

4. § 20.

Page 114

Exercise 9. In the cone let $O'A = 2R$, $O'C = R$, $DO' = x$, $AD = 2R - x$, and $DE = r$.

In the sphere $KH = 2R$, $FH = x$, $KF = 2R - x$, and $FG = r$.



In the cone $\frac{AD}{AO'} = \frac{DE}{O'C}$. Why? $\therefore \frac{2R - x}{2R} = \frac{r}{R}$. $\therefore r = ?$

In the sphere $\frac{KF}{FG} = \frac{FG}{FH}$ (§ 328a). $\therefore \frac{2R - x}{r} = \frac{r}{x}$, $r = ?$

Make the two values of r equal and solve for x .

Page 117

Proposition VI. 4. § 386.

8. § 348.

9. Substituting $\angle EAD$ for its equal, $\angle COB$, in 8.

Page 120

Proposition VII. 4. Reasons 1-3.

Page 121

Proposition VIII. 4. Axiom I. Substituting $B'D + DE$ for its equal, $B'E$.

6. Substituting $\angle A$ for its equal, DE , and $B'C'$ for its equal, $B'D + DC'$, in 4.

Page 122

Proposition IX. 2. Axiom I.

4. Axiom XII.

5. Otherwise there would be no triangle.

6. Reason 4.

Page 126

Proposition XI. 4. Reasons 1-3.

7. Reason 3.

Page 128

Proposition XII. 4. Axiom VII.

7. Reasons 2-6.

8. Axiom I.

9. Substituting the whole for the sum of its parts.

11. Evident from statements 9 and 10.

Page 130

Proposition XIII. 2. Axiom VIa.

4. Substituting $\triangle ADE$ for its equal $\triangle BFC$ in 2.

5. Axiom 1.

6. Also Axiom VIa.
7. Substituting 360° for its equal in 5.
8. Axiom IV.
9. Axiom II.
10. § 261.

Page 132

Proposition XIV. 4. The sides of one triangle are $\perp$ the sides of the other.

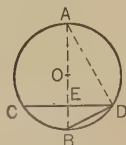
5. § 398.
7. Substituting $EF \times CD$ for its equal, $AB \times DH$, in 3.
10. Reason 4.
11. Reason 5.
12. Reason 6.
13. Substituting $EF \times CD$ for its equal, $AB \times DH$, in 9.

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Proposition XV. 4. Axiom I, and the whole may be substituted for the sum of its parts.

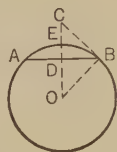
Page 138

Exercise 5. Show that $2 \pi r h = \pi(BD)^2$. Use § 328b.

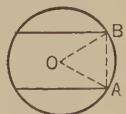


Exercise 6. We are to find the area of the zone ABE . Using the formula: Area $= 2 \pi r h$, $ED = h$. Hence we must find the value of ED .

$\triangle OBC$ is a right $\triangle$ with leg OB , hypotenuse OC , and OD the segment of the hypotenuse adjacent to OB . $\therefore \frac{OD}{OB} = \frac{OB}{OC}$ (§ 328b). Substitute the given values and find OD . Then $ED = OE - OD$, etc.



Exercise 8. $AB = h$, the altitude of the zone.



Page 142

Proposition XVI. 3. Axiom I. The whole may be substituted for the sum of its parts.

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Exercise 8. Try to use § 302.

Formulas of Mensuration for Solids

S = lateral area or area of curved surface ;

B, b = area of base of a solid ;

h = altitude ;

l = slant height ;

R, r = radius of circle or sphere

I. Areas of the surfaces of polyhedrons and the total surface of a cone or cylinder should be solved by finding the sum of the areas of the faces.

II. The following formulas should be memorized :

Prism and cylinder: $V = Bh.$

Pyramid and cone: $V = \frac{1}{3} Bh.$

Right circular cylinder: $S = 2\pi rh; V = \pi r^2 h.$

Right circular cone: $S = \pi rl; V = \frac{1}{3} \pi r^2 h.$

Sphere: $S = 4\pi r^2; V = \frac{4}{3} \pi r^3.$

Zone: $S = 2\pi rh.$

III. The use of the following formulas should be understood :

Frustum of a pyramid or cone: $V = \frac{1}{3} h(B + b + \sqrt{Bb}).$

Frustum of a right circular cone:

$$S = \pi l(R + r); V = \frac{1}{3} \pi h(R^2 + r^2 + Rr).$$

Spherical sector: $V = \frac{1}{3} rS$ (S = area of zone).

A *spherical segment* of one base should be treated as the difference between the spherical sector (spherical cone) and the cone.

TRIGONOMETRIC TABLE

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
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21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
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23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
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27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
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32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
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48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
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57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
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97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
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